

ASYMPTOTIC LINEAR STABILITY OF THE BENNEY-LUKE EQUATION IN 2D

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ABSTRACT. In this paper, we study transverse linear stability of line solitary waves to the 2-dimensional Benney-Luke equation which arises in the study of small amplitude long water waves in 3D. In the case where the surface tension is weak or negligible, we find a curve of resonant continuous eigenvalues near 0. Time evolution of these resonant continuous eigenmodes is described by a 1D damped wave equation in the transverse variable and it gives a linear approximation of the local phase shifts of modulating line solitary waves. In exponentially weighted space whose weight function increases in the direction of the motion of the line solitary wave, the other part of solutions to the linearized equation decays exponentially as $t \rightarrow \infty$.

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1. INTRODUCTION

In this paper, we study transverse linear stability of line solitary waves for the Benney-Luke equation

$$(1.1) \quad \partial_t^2 \Phi - \Delta \Phi + a \Delta^2 \Phi - b \Delta \partial_t^2 \Phi + (\partial_t \Phi)(\Delta \Phi) + \partial_t(|\nabla \Phi|^2) = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^2.$$

The Benney-Luke equation is an approximation model of small amplitude long water waves with finite depth originally derived by Benney and Luke [4] as a model for 3D water waves. Here $\Phi = \Phi(t, x, y)$ corresponds to a velocity potential of water waves. We remark that (1.1) is an isotropic model for propagation of water waves whereas KdV, BBM and KP equations are unidirectional models. See e.g. [6, 7] for the other bidirectional models of 2D and 3D water waves.

The parameters a, b are positive and satisfy $a - b = \hat{\tau} - 1/3$, where $\hat{\tau}$ is the inverse Bond number. If we think of waves propagating in one direction, slowly evolving in time and having weak transverse variation, then the Benney-Luke equation can be formally reduced to the KP-II equation if $0 < a < b$ and to the KP-I equation if $a > b > 0$. More precisely, the Benney-Luke equation (1.1) is reduced to

$$2f_{\tilde{x}\tilde{t}} + (b - a)f_{\tilde{x}\tilde{x}\tilde{x}} + 3f_{\tilde{x}}f_{\tilde{x}\tilde{x}} + f_{\tilde{y}\tilde{y}} = 0$$

in the coordinate $\tilde{t} = \epsilon^3 t$, $\tilde{x} = \epsilon(x - t)$ and $\tilde{y} = \epsilon^2 y$ by taking terms only of order ϵ^5 , where $\Phi(t, x, y) = \epsilon f(\tilde{t}, \tilde{x}, \tilde{y})$. See e.g. [22] for the details. In this paper, we will assume $0 < a < b$, which corresponds to the case where the surface tension is weak or negligible.

The solution $\Phi(t)$ of the Benney-Luke equation (1.1) formally satisfies the energy conservation law

$$(1.2) \quad E(\Phi(t), \partial_t \Phi(t)) = E(\Phi_0, \Psi_0) \quad \text{for } t \in \mathbb{R},$$

where

$$E(\Phi, \Psi) := \int_{\mathbb{R}^2} \{ |\nabla \Phi|^2 + a(\Delta \Phi)^2 + \Psi^2 + b|\nabla \Psi|^2 \} dx dy,$$

and (1.1) is globally well-posed in the energy class $(\dot{H}^2(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2)) \times H^1(\mathbb{R}^2)$ (see [40]). The Benney Luke equation (1.1) has a 3-parameter family of line solitary wave solutions

$$(1.3) \quad \Phi(t, x, y) = \varphi_c(x \cos \theta + y \sin \theta - ct + \gamma), \quad \pm c > 1, \quad \gamma \in \mathbb{R}, \quad \theta \in [0, 2\pi),$$

where

$$\varphi_c(x) = \frac{2(c^2 - 1)}{c\alpha_c} \tanh\left(\frac{\alpha_c}{2}x\right), \quad \alpha_c = \sqrt{\frac{c^2 - 1}{bc^2 - a}},$$

and

$$q_c(x) := \varphi'_c(x) = \frac{c^2 - 1}{c} \operatorname{sech}^2\left(\frac{\alpha_c x}{2}\right)$$

is a solution of

$$(1.4) \quad (bc^2 - a)q_c'' - (c^2 - 1)q_c + \frac{3c}{2}q_c^2 = 0.$$

Stability of solitary waves to the 1-dimensional Benney-Luke equation are studied by [38] for the strong surface tension case $a > b > 0$ and by [30] for the weak surface tension case $b > a > 0$. If $a > b > 0$, then (1.1) has a stable ground state for c satisfying $0 < c^2 < 1$

([33, 39]). See also [23] for the algebraic decay property of the ground state. In view of [42, 43], line solitary waves for the 2-dimensional Benney-Luke equation are expected to be unstable in this parameter regime. On the other hand if $0 < a < b$ and $c := \sqrt{1 + \epsilon^2}$ is close to 1 (the sonic speed), then $\varphi_c(x - ct)$ is expected to be transversally stable because $q_c(x)$ is similar to a KdV 1-soliton and line solitons of the KP-II equation is transversally stable ([21, 27, 28]).

The dispersion relation for the linearization of (1.1) around 0 is

$$\omega^2 = (\xi^2 + \eta^2) \frac{1 + a(\xi^2 + \eta^2)}{1 + b(\xi^2 + \eta^2)}$$

for a plane wave solution $\Phi(t, x, y) = e^{i(x\xi + y\eta - \omega t)}$. If $b > a > 0$, then $|\nabla\omega| \leq 1$ and line solitary waves travel faster than the maximum group velocity of linear waves. Measuring the size of perturbations with an exponentially weighted norm biased in the direction of motion of a line solitary wave, we can observe that perturbations which are decoupled from the line solitary wave decay as $t \rightarrow \infty$. In the 1-dimensional case, small solitary waves are exponentially linearly stable in the weighted space and $\lambda = 0$ is an isolated eigenvalue of the linearized operator (see [30]). In our problem, however, the value $\lambda = 0$ is not an isolated eigenvalue. This is because line solitary waves do not decay in the transverse direction. Indeed, for any size of line solitary waves of (1.1), there appears a curve of continuous spectrum that goes through $\lambda = 0$ and locates in the stable half plane (Theorem 2.1). The curve of continuous eigenvalues has to do with perturbations that propagate toward the transverse direction along the crest of the line solitary wave (Theorem 2.3). If line solitary waves are small, the rest of the spectrum locates in a stable half plane $\{\lambda \in \mathbb{C} \mid \Re\lambda \leq -\beta < 0\}$ (Theorem 2.4). For the KP-II equation, the spectrum of the linearized operator around a 1-line soliton near $\lambda = 0$ can be obtained explicitly thanks to the integrability of the equation (see [2, 9, 28]). In this paper, we will use the Lyapunov-Schmidt method to find resonant eigemodes of the linearized operator.

To prove non-existence of unstable modes for the linearized operator around small line solitary waves, we make use of the KP-II approximation of the the linearized operator of (1.1) on long length scales and make use of the transverse linear stability of line solitons for the KP-II equation. For 1-dimensional long wave models, non-existence of unstable modes for the linearized operator around solitary waves has been proved by utilizing spectral stability of KdV solitons. See e.g. [12, 27, 24, 34, 36] and [30] for the 1-dimensional Benney-Luke equation. We expect that the KP-II approximation of the linearized operator is useful to other 2-dimensional long wave models such as KP-BBM and Boussinesq systems with no surface tension (see e.g. [10]).

Now let us introduce several notations. For an operator A , we denote by $\sigma(A)$ the spectrum and by $D(A)$ and $R(A)$ the domain and the range of the operator A , respectively. For Banach spaces V and W , let $B(V, W)$ be the space of all linear continuous operators from V to W and $\|T\|_{B(V, W)} = \sup_{\|x\|_V=1} \|Tu\|_W$ for $T \in B(V, W)$. We abbreviate $B(V, V)$ as $B(V)$. For

$f \in \mathcal{S}(\mathbb{R}^n)$ and $m \in \mathcal{S}'(\mathbb{R}^n)$, let

$$\begin{aligned} (\mathcal{F}f)(\xi) &= \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix\xi} dx, \\ (\mathcal{F}^{-1}f)(x) &= \check{f}(x) = \hat{f}(-x), \end{aligned}$$

and $(m(D)f)(x) = (2\pi)^{-n/2}(\check{m} * f)(x)$. We denote $\langle f, g \rangle$ by

$$\langle f, g \rangle = \sum_{j=1}^m \int_{\mathbb{R}} f_j(x) \overline{g_j(x)} dx$$

for \mathbb{C}^m -valued functions $f = (f_1, \dots, f_m)$ and $g = (g_1, \dots, g_m)$.

Let $L_\alpha^2(\mathbb{R}^2) = L^2(\mathbb{R}^2; e^{2\alpha x} dx dy)$, $L_\alpha^2(\mathbb{R}) = L^2(\mathbb{R}; e^{2\alpha x} dx)$ and let $H_\alpha^k(\mathbb{R}^2)$ and $H_\alpha^k(\mathbb{R})$ be Hilbert spaces with the norms

$$\begin{aligned} \|u\|_{H_\alpha^k(\mathbb{R}^2)} &= \left(\|\partial_x^k u\|_{L_\alpha^2(\mathbb{R}^2)}^2 + \|\partial_y^k u\|_{L_\alpha^2(\mathbb{R}^2)}^2 + \|u\|_{L_\alpha^2(\mathbb{R}^2)}^2 \right)^{1/2}, \\ \|u\|_{H_\alpha^k(\mathbb{R})} &= \left(\|\partial_x^k u\|_{L_\alpha^2(\mathbb{R})}^2 + \|u\|_{L_\alpha^2(\mathbb{R})}^2 \right)^{1/2}. \end{aligned}$$

We use $a \lesssim b$ and $a = O(b)$ to mean that there exists a positive constant such that $a \leq Cb$. Various constants will be simply denoted by C and C_i ($i \in \mathbb{N}$) in the course of the calculations. We denote $\langle x \rangle = \sqrt{1 + x^2}$ for $x \in \mathbb{R}$.

2. STATEMENT OF THE RESULT

Since (1.1) is isotropic and translation invariant, we may assume $\theta = \gamma = 0$ in (1.3) without loss of generality. Let $\Psi = \partial_t \Phi$, $A = I - a\Delta$ and $B = I - b\Delta$. Then in the moving coordinate $z = x - ct$, the Benney-Luke equation (1.1) can be rewritten as

$$(2.1) \quad \begin{cases} \partial_t \Phi = c\partial_z \Phi + \Psi, \\ \partial_t \Psi = c\partial_z \Psi + B^{-1}A\Delta\Phi - B^{-1}(\Psi\Delta\Phi + 2\nabla\Phi \cdot \nabla\Psi), \end{cases}$$

Let $r_c(z) = -cq_c(z)$. Linearizing (2.1) around $(\Phi, \Psi) = (\varphi_c(z), r_c(z))$, we have

$$(2.2) \quad \begin{aligned} \partial_t \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} &= \mathcal{L} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}, \\ \mathcal{L} &= \mathcal{L}_0 + V, \quad \mathcal{L}_0 = \begin{pmatrix} c\partial_z & 1 \\ B^{-1}A\Delta & c\partial_z \end{pmatrix}, \end{aligned}$$

$$(2.3) \quad V = -B^{-1} \begin{pmatrix} 0 & 0 \\ v_{1,c} & v_{2,c} \end{pmatrix}, \quad v_{1,c} = 2r'_c(z)\partial_z + r_c(z)\Delta, \quad v_{2,c} = 2q_c(z)\partial_z + q'_c(z).$$

We study linear stability of (2.2) in a weighted space $X := H_\alpha^1(\mathbb{R}^2) \times L_\alpha^2(\mathbb{R}^2)$. Let $\mathcal{L}(\eta)u(z) = e^{-iy\eta}\mathcal{L}(e^{iy\eta}u(z))$ for $\eta \in \mathbb{R}$. Note that V is independent of y . For each small $\eta \neq 0$, the operator $\mathcal{L}(\eta)$ has two stable eigenvalues.

Theorem 2.1. *Let $0 < a < b$ and $k \in \mathbb{N}$. Fix $c > 1$ and $\alpha \in (0, \alpha_c)$. Then there exist a positive constant η_0 , $\lambda(\eta) \in C^\infty([-\eta_0, \eta_0])$,*

$$\zeta(\cdot, \eta) \in C^\infty([-\eta_0, \eta_0]; H_\alpha^k(\mathbb{R}) \times H_\alpha^{k-1}(\mathbb{R})), \quad \zeta^*(\cdot, \eta) \in C^\infty([-\eta_0, \eta_0]; H_{-\alpha}^k(\mathbb{R}) \times H_{-\alpha}^{k-1}(\mathbb{R}))$$

such that

$$(2.4) \quad \begin{aligned} \mathcal{L}(\eta)\zeta(z, \pm\eta) &= \lambda(\pm\eta)\zeta(z, \pm\eta), \quad \mathcal{L}(\eta)^*\zeta^*(z, \pm\eta) = \lambda(\mp\eta)\zeta^*(z, \pm\eta), \\ \lambda(\eta) &= i\lambda_1\eta - \lambda_2\eta^2 + O(\eta^3), \end{aligned}$$

$$(2.5) \quad \zeta(\cdot, \eta) = \zeta_1 + i\lambda_1\eta\zeta_2 + O(\eta^2) \quad \text{in } H_\alpha^k(\mathbb{R}) \times H_\alpha^{k-1}(\mathbb{R}),$$

$$(2.6) \quad \zeta^*(\cdot, \eta) = \zeta_2^* - i\lambda_1\eta\zeta_1^* + O(\eta^2) \quad \text{in } H_{-\alpha}^k(\mathbb{R}) \times H_{-\alpha}^{k-1}(\mathbb{R}),$$

$$(2.7) \quad \overline{\lambda(\eta)} = \lambda(-\eta), \quad \overline{\zeta(z, \eta)} = \zeta(z, -\eta), \quad \overline{\zeta^*(z, \eta)} = \zeta^*(z, -\eta) \quad \text{for } \eta \in [-\eta_0, \eta_0] \text{ and } z \in \mathbb{R},$$

where λ_1 and λ_2 are positive constants, $A_0 = 1 - a\partial_z^2$, $B_0 = 1 - b\partial_z^2$ and

$$\begin{aligned} \zeta_1 &= \begin{pmatrix} q_c \\ r'_c \end{pmatrix}, \quad \zeta_2 = \begin{pmatrix} \int_z^\infty \partial_c q_c \\ -\partial_c r_c \end{pmatrix}, \\ \zeta_1^* &= c \begin{pmatrix} -B_0\partial_c r_c - 2q_c\partial_c q_c - q'_c \int_{-\infty}^z \partial_c q_c \\ B_0 \int_{-\infty}^z \partial_c q_c \end{pmatrix}, \quad \zeta_2^* = \begin{pmatrix} A_0 q'_c \\ -B_0 r_c \end{pmatrix}. \end{aligned}$$

Remark 2.1. We remark that $\mathcal{L}(0)$ is a linearized operator of the 1-dimensional Benney-Luke equation around $\varphi_c(z)$ and ζ_1 and ζ_2 belong to the generalized kernel of $\mathcal{L}(0)$. More precisely,

$$\begin{aligned} \mathcal{L}(0)\zeta_1 &= 0, \quad \mathcal{L}(0)\zeta_2 = \zeta_1, \quad \mathcal{L}(0)^*\zeta_1^* = \zeta_2^*, \quad \mathcal{L}(0)^*\zeta_2^* = 0, \\ \ker_g(\mathcal{L}(0)) &= \text{span}\{\zeta_1, \zeta_2\}, \quad \ker_g(\mathcal{L}(0)) = \text{span}\{\zeta_1^*, \zeta_2^*\}. \end{aligned}$$

The eigenvalue $\lambda = 0$ for $\mathcal{L}(0)$ splits into two stable eigenvalues $\lambda(\pm\eta)$ for $\mathcal{L}(\eta)$ with $\eta \neq 0$.

In the exponentially weighted space $L_\alpha^2(\mathbb{R})$, the value $\lambda = 0$ is an isolated eigenvalue of $\mathcal{L}(0)$ and there exists a $\beta > 0$ such that

$$\sigma(\mathcal{L}(0)) \setminus \{0\} \subset \{\lambda \in \mathbb{C} \mid \Re \lambda \leq -\beta\}$$

provided $c > 1$ and c is sufficiently close to 1. See Lemma 2.1, Theorem 2.3 and Appendix B in [30].

Remark 2.2. We expect that $\zeta_k(\cdot, \eta)$ and $\zeta_k^+(\cdot, \eta)$ ($k = 1, 2$) do not belong to $L^2(\mathbb{R})$ as is the same with continuous resonant modes for the KP-II equation. This is a reason why we study spectral stability of \mathcal{L} in the exponentially weighted space X .

We will prove Theorem 2.1 by using the Lyapunov Schmidt method in Section 6.

Let $\mathcal{P}(\eta_0)$ be the spectral projection onto the subspace corresponding to the continuous eigenvalues $\{\lambda(\eta)\}_{-\eta_0 \leq \eta \leq \eta_0}$ and $\mathcal{Q}(\eta_0) = I - \mathcal{P}(\eta_0)$. Let $Z = \mathcal{Q}(\eta_0)(H_\alpha^1(\mathbb{R}^2) \times L_\alpha^2(\mathbb{R}^2))$. If \mathcal{L} is spectrally stable, then $e^{t\mathcal{L}}|_Z$ is exponentially stable.

Corollary 2.2. *Let $0 < a < b$, $c > 1$ and $\alpha \in (0, \alpha_c)$. Consider the operator \mathcal{L} in the space $X = H_\alpha^1(\mathbb{R}^2) \times L_\alpha^2(\mathbb{R}^2)$. Assume that there exist positive constants β and η_0 such that*

$$(H) \quad \sigma(\mathcal{L}|_Z) \subset \{\lambda \mid \Re \lambda \leq -\beta\},$$

where $\mathcal{L}|_Z$ is the restriction of the operator \mathcal{L} on Z . Then for any $\beta' < \beta$, there exists a positive constant C such that

$$(2.8) \quad \|e^{t\mathcal{L}}\mathcal{Q}(\eta_0)\|_{B(X)} \leq Ce^{-\beta't} \quad \text{for any } t \geq 0.$$

The semigroup estimate (2.8) follows from the assumption (H) and the Geahart-Prüss theorem [15, 37] which tells us that the boundedness of C^0 -semigroup in a Hilbert space is equivalent to the uniform boundedness of the resolvent operator on the right half plane. See also [17, 18].

Time evolution of the continuous eigenmodes $\{e^{t\lambda(\eta)}g(z, \eta)\}_{-\eta_0 \leq \eta \leq \eta_0}$ can be considered as a linear approximation of non-uniform phase shifts of modulating line solitary waves. For the KP-II equation, modulations of the local amplitude and the angle of the local phase shift of a line soliton are described by a system of Burgers' equations (see [28, Theorems 1.4 and 1.5]). In this paper, we find the first order asymptotics of solutions for the linearized equation (2.2) is described by a wave equation with a diffraction term and it tends to a constant multiple of the x -derivative of the line solitary wave as $t \rightarrow \infty$.

Theorem 2.3. *Let $0 < a < b$, $c > 1$, α be as in Theorem 2.2 and $(\Phi_0, \Psi_0) \in H_\alpha^2(\mathbb{R}^2) \times H_\alpha^1(\mathbb{R}^2)$. Assume (H). Then a solution of (2.2) with $(\Phi(0), \partial_t \Phi(0)) = (\Phi_0, \Psi_0)$ satisfies*

$$\left\| \begin{pmatrix} \partial_z \Phi(t, z, y) \\ \partial_t \Phi(t, z, y) \end{pmatrix} - (H_t * W_t * f)(y) \begin{pmatrix} q'_c(z) \\ r'_c(z) \end{pmatrix} \right\|_{L_\alpha^2(\mathbb{R}_z) L^\infty(\mathbb{R}_y)} = O(t^{-1/4}) \quad \text{as } t \rightarrow \infty,$$

where $f(y) = \langle cB_0\Psi_0 - A_0\partial_z\Phi_0, q_c \rangle$, $H_t(y) = (4\pi\lambda_2 t)^{-1/2} e^{-y^2/4\lambda_2 t}$, $\kappa_1 = \frac{\lambda_1}{2} \frac{d}{dc} E(q_c, r_c)$ and $W_t(y) = (2\kappa_1)^{-1}$ for $y \in [-\lambda_1 t, \lambda_1 t]$ and $W_t(y) = 0$ otherwise.

We remark that if $f(y)$ is well localized and $\int_{\mathbb{R}} f(y) dy \neq 0$, then $H_t * W_t * f(y) \simeq (2\kappa_1)^{-1} \int_{\mathbb{R}} f(y) dy$ on any compact intervals in y as $t \rightarrow \infty$. The first order asymptotics of solutions to (2.2) suggests that the local phase shift of line solitary waves propagate mostly at constant speed toward $y = \pm\infty$.

If c is close to 1, then the assumption (H) is valid and the spectrum of \mathcal{L} is similar to that of the linearized KP-II operator around a line soliton. To utilize the spectral property of the linearized operators of the KP-II equation around 1-line solitons, we introduce the scaled parameters and variables

$$(2.9) \quad \lambda = \epsilon^3 \Lambda, \quad c^2 = 1 + \epsilon^2, \quad \hat{z} = \epsilon z, \quad \hat{y} = \epsilon^2 y, \quad \xi = \epsilon \hat{\xi}, \quad \eta = \epsilon^2 \hat{\eta},$$

and translate the solitary wave profile $q_c(x)$ as

$$(2.10) \quad q_c(z) = \epsilon^2 \theta_\epsilon(\hat{z}), \quad \theta_\epsilon(\hat{z}) = \frac{1}{c} \operatorname{sech}^2 \left(\frac{\hat{\alpha}_\epsilon \hat{z}}{2} \right), \quad \hat{\alpha}_\epsilon = \frac{1}{\sqrt{bc^2 - a}}.$$

Let

$$\begin{aligned} \hat{\alpha}_0 &= (b - a)^{-1/2}, \quad \theta_0(\hat{z}) = \operatorname{sech}^2 \left(\frac{\hat{\alpha}_0}{2} \hat{z} \right), \\ \mathcal{L}_{KP} &= -\frac{1}{2} \{ (b - a) \partial_{\hat{z}}^3 - \partial_{\hat{z}} + \partial_{\hat{z}}^{-1} \partial_{\hat{y}}^2 + 3 \partial_{\hat{z}} (\theta_0 \cdot) \}. \end{aligned}$$

We remark that the operator \mathcal{L}_{KP} is the linearization of the KP-II equation

$$(2.11) \quad 2\partial_t u + (b-a)\partial_x^3 u + \partial_x^{-1}\partial_y^2 u + \frac{3}{2}\partial_x(u^2) = 0$$

around its line soliton solution $\theta_0(x-t)$. The linearized operator \mathcal{L}_{KP} has continuous eigenvalues $\lambda_{KP}(\eta) = \frac{i\eta}{\sqrt{3}}\sqrt{1+i\gamma_1\eta}$ which has to do with dynamics of modulating line solitons (see [9, 28] and Section 3.1).

In the low frequency regime, we can deduce the eigenvalue problem

$$(2.12) \quad \mathcal{L} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

to $\mathcal{L}_{KP}\partial_z u = \Lambda\partial_z u$ provided ϵ is sufficiently small. More precisely, we have the following.

Theorem 2.4. *Let $c = \sqrt{1+\epsilon^2}$, $\alpha = \hat{\alpha}\epsilon$ and $\hat{\alpha} \in (0, \hat{\alpha}_0/2)$. Then there exist positive constants $\epsilon_0, \eta_0, \hat{\beta}$ and a smooth function $\lambda_\epsilon(\eta)$ such that if $\epsilon \in (0, \epsilon_0)$, then*

$$(2.13) \quad \sigma(\mathcal{L}) \setminus \{\lambda_\epsilon(\eta) \mid \eta \in [-\epsilon^2\eta_0, \epsilon^2\eta_0]\} \subset \{\lambda \in \mathbb{C} \mid \Re \lambda \leq -\hat{\beta}\epsilon^3\},$$

$$(2.14) \quad \lim_{\epsilon \downarrow 0} |\epsilon^{-3}\lambda_\epsilon(\epsilon^2\eta) - \lambda_{KP}(\eta)| = O(\eta^3) \quad \text{for } \eta \in [-\eta_0, \eta_0],$$

$$(2.15) \quad \|e^{t\mathcal{L}}\mathcal{Q}(\epsilon^2\eta_0)\|_{B(X)} \leq Ke^{-\hat{\beta}\epsilon^3 t} \quad \text{for any } t \geq 0,$$

where K is a constant that does not depend on t .

3. RESONANT MODES OF THE LINEARIZED OPERATOR

In this section, we will prove the existence of resonant continuous eigenvalues of \mathcal{L} near $\lambda = 0$ and show that the resonant eigenvalues and resonant eigenmodes for \mathcal{L} are similar to those for the linearized KP-II operator \mathcal{L}_{KP} provided line solitary waves are small.

3.1. Spectral stability in the KP-II scaling regime. First, we recall some spectral properties of the linearized KP-II equation around 1-line solitons. Let us consider the eigenvalue problem of the linearized operator of (2.11) around θ_0 . Let

$$\begin{aligned} \mathcal{L}_{KP,0} &= -\frac{1}{2}\{(b-a)\partial_z^3 - \partial_z + \partial_z^{-1}\partial_y^2\}, \quad \mathcal{L}_{KP} = \mathcal{L}_{KP,0} - \frac{3}{2}\partial_z(\theta_0 \cdot), \\ \mathcal{L}_{KP}(\eta) &= -\frac{1}{2}\partial_z\{(b-a)\partial_z^2 - 1 + 3\theta_0\} + \frac{\eta^2}{2}\partial_z^{-1}. \end{aligned}$$

Formally, we have $\mathcal{L}_{KP}(u(z)e^{iy\eta}) = e^{iy\eta}(\mathcal{L}_{KP}(\eta)u)(z)$. The operator $\mathcal{L}_{KP,0}$ is spectrally stable in exponentially weighted spaces.

Lemma 3.1. *Let $\hat{\alpha} \in (0, \hat{\alpha}_0)$ and $\hat{\beta}_0 = \frac{\hat{\alpha}}{2}\{1 - (b-a)\hat{\alpha}^2\}$. Then*

$$(3.1) \quad \|(\Lambda - \mathcal{L}_{KP,0})^{-1}\|_{B(L_\alpha^2(\mathbb{R}^2))} \leq (\Re \Lambda + \hat{\beta}_0)^{-1} \quad \text{for } \Lambda \text{ satisfying } \Re \Lambda > -\hat{\beta}_0.$$

Moreover, there exists a positive constant C such that if $\Re \Lambda > -\hat{\beta}_0$,

$$(3.2) \quad \|\partial_z^j (\Lambda - \mathcal{L}_{KP,0})^{-1}\|_{B(L_\alpha^2(\mathbb{R}^2))} \leq C \left(\Re \Lambda + \frac{\hat{\beta}_0}{2} \right)^{-1+\frac{j}{2}} \quad \text{for } j = 1, 2,$$

$$(3.3) \quad \|(\Lambda - \mathcal{L}_{KP,0})^{-1}\|_{B(L_\alpha^2(\mathbb{R}^2))} \leq C \left| \Lambda + \frac{\hat{\beta}_0}{2} \right|^{-2/3}.$$

Proof. By the Plancherel theorem,

$$(3.4) \quad \|g\|_{L_\alpha^2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} e^{2\alpha x} |g(x, y)|^2 dx dy = \int_{\mathbb{R}^2} |\hat{g}(\xi + i\alpha, \eta)|^2 d\xi d\eta$$

for any $g \in C_0(\mathbb{R}^2)$ and an operator $m(D) := \frac{1}{2\pi} \check{m} * f$ is bounded on $L_\alpha^2(\mathbb{R}^2)$ if and only if

$$(3.5) \quad \|m(D)\|_{B(L_\alpha^2(\mathbb{R}^2))} = \sup_{(\xi, \eta) \in \mathbb{R}^2} |m(\xi + i\alpha, \eta)| < \infty.$$

Suppose $f \in L_\alpha^2(\mathbb{R}^2)$ and that u is a solution of

$$(\Lambda - \mathcal{L}_{KP,0})u = f.$$

Then

$$\hat{u}(\xi, \eta) = \frac{\hat{f}(\xi, \eta)}{\Lambda - \mathcal{L}_{KP,0}(\xi, \eta)},$$

where $\mathcal{L}_{KP,0}(\xi, \eta) = \frac{i}{2} \{(b-a)\xi^3 + \xi - \xi^{-1}\eta^2\}$. Since

$$\Re \mathcal{L}_{KP,0}(\xi + i\hat{\alpha}, \eta) = -\frac{1}{2} \left\{ 3(b-a)\hat{\alpha}\xi^2 + 2\hat{\beta}_0 + \frac{\hat{\alpha}\eta^2}{\xi^2 + \hat{\alpha}^2} \right\} \geq -\hat{\beta}_0,$$

it follows from (3.5) that for $j = 0, 1, 2$ and Λ with $\Re \Lambda > -\hat{\beta}_0$,

$$\|\partial_z^j (\Lambda - \mathcal{L}_{KP,0})^{-1}\|_{B(L_\alpha^2(\mathbb{R}^2))} = \sup_{(\xi, \eta) \in \mathbb{R}^2} \frac{|\xi + i\hat{\alpha}|^j}{|\Lambda - \mathcal{L}_{KP,0}(\xi + i\hat{\alpha}, \eta)|}.$$

Thus we have (3.1) and (3.2). Moreover, we have (3.3) because $|\Im \mathcal{L}_{KP,0}(\xi + i\hat{\alpha}, \eta)| \lesssim \{-\Re \mathcal{L}_{KP,0}(\xi + i\hat{\alpha}, \eta)\}^{3/2}$. \square

Let $\gamma_1 = 4\sqrt{(b-a)/3}$, $\hat{x} = \frac{\hat{\alpha}_0}{2}x$ and

$$\begin{aligned} \lambda_{KP}(\eta) &= \frac{i\eta}{\sqrt{3}} \sqrt{1 + i\gamma_1\eta}, \\ g_0(x, \eta) &= \frac{2(b-a)}{\gamma_1\sqrt{1 + i\gamma_1\eta}} \partial_x^2 \left(e^{-\sqrt{1+i\gamma_1\eta}\hat{x}} \operatorname{sech} \hat{x} \right), \\ g_0^*(x, \eta) &= \frac{i}{\eta} \partial_x \left(e^{\sqrt{1-i\gamma_1\eta}\hat{x}} \operatorname{sech} \hat{x} \right). \end{aligned}$$

Using Lemma 2.1 in [28] and the change of variable

$$x \mapsto \frac{\hat{\alpha}_0}{2}x, \quad y \mapsto \frac{1}{\gamma_1}y, \quad \eta \mapsto \gamma_1\eta,$$

we have for $\eta \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned}\mathcal{L}_{KP}(\eta)g_0(x, \pm\eta) &= \lambda_{KP}(\pm\eta)g_0(x, \pm\eta), \\ \mathcal{L}_{KP}(\eta)^*g_0^*(x, \pm\eta) &= \lambda_{KP}(\mp\eta)g_0^*(x, \pm\eta), \\ \int_{\mathbb{R}} g_0(x, \eta)\overline{g_0^*(x, \eta)} dx &= 1, \quad \int_{\mathbb{R}} g_0(x, \eta)\overline{g_0^*(x, -\eta)} dx = 0.\end{aligned}$$

To resolve the singularity of $g_0(x, \eta)$ and the degeneracy of $g_0^*(x, \eta)$ at $\eta = 0$, we decompose them into their real parts and imaginary parts. Let

$$\begin{aligned}g_{0,1}(x, \eta) &= g_0(x, \eta) + g_0(x, -\eta), \quad g_{0,2}(x, \eta) = \frac{1}{i\eta}\{g_0(x, \eta) - g_0(x, -\eta)\}, \\ g_{0,1}^*(x, \eta) &= \frac{1}{2}\{g_0^*(x, \eta) + g_0^*(x, -\eta)\}, \quad g_{0,2}^*(x, \eta) = \frac{\eta}{2i}\{g_0^*(x, \eta) - g_0^*(x, -\eta)\}.\end{aligned}$$

Then

$$\int_{\mathbb{R}} g_{0,j}(x, \eta)\overline{g_{0,k}^*(x, \eta)} dx = \delta_{jk} \quad \text{for } j, k = 1, 2.$$

Moreover, we see that $g_{0,k}(x, \eta)$ and $g_{0,k}^*(x, \eta)$ are even in η and that for $k = 1, 2$ and $\hat{\alpha} \in (0, \hat{\alpha}_0)$,

$$(3.6) \quad \|g_{0,k}(\cdot, \eta) - g_{0,k}(\cdot, 0)\|_{L_{\hat{\alpha}}^2} + \|g_{0,k}^*(\cdot, \eta) - g_{0,k}^*(\cdot, 0)\|_{L_{-\hat{\alpha}}^2} = O(\eta^2),$$

$$(3.7) \quad g_{0,1}(x, 0) = -\frac{\sqrt{3}}{2}\theta_0'(x), \quad g_{0,2}(x, 0) = \theta_0(x) + \left(\frac{x}{2} + \hat{\alpha}_0^{-1}\right)\theta_0'(x),$$

$$(3.8) \quad g_{0,1}^*(x, 0) = \frac{\hat{\alpha}_0}{2\sqrt{3}} \int_{-\infty}^x (x_1\theta_0'(x_1) + 2\theta_0(x_1)) dx_1, \quad g_{0,2}^*(x, 0) = \frac{\hat{\alpha}_0}{2}\theta_0(x).$$

Let $\mathcal{P}_{KP}(\eta_0)$ be the spectral projection to resonant modes $\{g_0(x, \pm\eta)e^{iy\eta}\}_{-\eta_0 \leq \eta \leq \eta_0}$ defined by

$$\begin{aligned}\mathcal{P}_{KP}(\eta_0)f(x, y) &= \frac{1}{\sqrt{2\pi}} \sum_{k=1,2} \int_{-\eta_0}^{\eta_0} a_{0,k}(\eta)g_{0,k}(x, \eta)e^{iy\eta} d\eta, \\ a_{0,k}(\eta) &= \int_{\mathbb{R}} (\mathcal{F}_y f)(x, \eta) \cdot g_{0,k}^*(x, \eta) dx,\end{aligned}$$

and let $\mathcal{Q}_{KP}(\eta_0) = I - \mathcal{P}_{KP}(\eta_0)$. By Lemma 3.1 in [28], the operator $\mathcal{P}_{KP}(\eta_0)$ and $\mathcal{Q}_{KP}(\eta_0)$ are bounded on $L_{\hat{\alpha}}^2(\mathbb{R}^2)$ for $\hat{\alpha} \in (0, \hat{\alpha}_0)$. Moreover, we have the following.

Proposition 3.2. *Let $\hat{\alpha} \in (0, \hat{\alpha}_0)$ and η_* be a positive number satisfying $\frac{\hat{\alpha}}{2}(\Re\sqrt{1+i\gamma\eta_*}-1) = \hat{\alpha}$. For any $\eta_0 \in (0, \eta_*)$, there exists a positive number b such that*

$$\sup_{\Re\Lambda \geq -b} \|(\Lambda - \mathcal{L}_{KP})^{-1}\mathcal{Q}_{KP}(\eta_0)\|_{B(L_{\hat{\alpha}}^2(\mathbb{R}^2))} < \infty.$$

Proof. By Proposition 3.2 in [28], there exist positive constants b_1 and C such that

$$\|e^{t\mathcal{L}_{KP}}\mathcal{Q}_{KP}(\eta_0)\|_{B(L_{\hat{\alpha}}^2(\mathbb{R}^2))} \leq Ce^{-b_1 t}.$$

If $\Re\Lambda \geq -b > -b_1$, then

$$\|(\Lambda - \mathcal{L}_{KP})^{-1}\mathcal{Q}_{KP}(\eta_0)\|_{B(L_{\hat{\alpha}}^2)} \leq \int_0^\infty \|e^{-\Lambda t}e^{t\mathcal{L}_{KP}}\mathcal{Q}_{KP}(\eta_0)\|_{B(L_{\hat{\alpha}}^2(\mathbb{R}^2))} dt \lesssim \frac{1}{b_1 - b}.$$

□

3.2. Resonant modes. In this subsection, we will prove the existence of continuous resonant modes of \mathcal{L} near $\lambda = 0$ by using the Lyapunov Schmidt method. Let

$$\begin{aligned} A(\eta) &= 1 + a\eta^2 - a\partial_z^2, \quad B(\eta) = 1 + b\eta^2 - b\partial_z^2, \\ \mathcal{L}_0(\eta) &= \begin{pmatrix} c\partial_z & 1 \\ B(\eta)^{-1}A(\eta)(\partial_z^2 - \eta^2) & c\partial_z \end{pmatrix}, \\ \mathcal{L}(\eta) &= \mathcal{L}_0(\eta) + V(\eta), \quad V(\eta) = -B(\eta)^{-1} \begin{pmatrix} 0 & 0 \\ v_{1,c}(\eta) & v_{2,c}(\eta) \end{pmatrix}, \\ v_{1,c}(\eta) &= 2r'_c\partial_z + r_c(\partial_z^2 - \eta^2), \quad v_{2,c}(\eta) = 2q'_c\partial_z + q'_c. \end{aligned}$$

If $e^{iy\eta}(u_1(z), u_2(z))$ is a solution of (2.12), then

$$(3.9) \quad \mathcal{L}(\eta) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

or equivalently,

$$(3.10) \quad \{A(\eta)(\partial_z^2 - \eta^2) - (\lambda - c\partial_z)^2 B(\eta)\}u_1 - v_{1,c}(\eta)u_1 - v_{2,c}(\eta)(\lambda - c\partial_z)u_1 = 0,$$

$$(3.11) \quad u_2 = (\lambda - c\partial_z)u_1.$$

We will find solutions of (3.9) in $H_\alpha^1(\mathbb{R}) \times L_\alpha^2(\mathbb{R})$ for small η . Using the change of variables (2.9) and (2.10) and dropping the hats in the resulting equation, we have

$$(3.12) \quad F(U, \Lambda, \epsilon, \eta) := 2L_\epsilon(\eta)U - \Lambda T_1(\epsilon, \eta)U + \epsilon^2 \Lambda^2 B_\epsilon(\eta) \partial_z^{-1} U = 0,$$

where $U(z) = \partial_z u_1(z/\epsilon)$ and

$$\begin{aligned} L_\epsilon(\eta) &= -\frac{1}{2}\partial_z \{(bc^2 - a)\partial_z^2 - 1 + 3c\theta_\epsilon\} + \frac{\eta^2}{2}T_2(\epsilon, \eta), \\ T_1(\epsilon, \eta) &= 2cB_\epsilon(\eta) - \epsilon^2(2\theta_\epsilon + \theta'_\epsilon\partial_z^{-1}), \quad T_2(\epsilon, \eta) = \{A_\epsilon(\eta) + \epsilon^2(bc^2 - a)\partial_z^2 + c\epsilon^2\theta_\epsilon\}\partial_z^{-1}, \\ A_\epsilon(\eta) &= 1 + a\epsilon^2(\epsilon^2\eta^2 - \partial_z^2), \quad B_\epsilon(\eta) = 1 + b\epsilon^2(\epsilon^2\eta^2 - \partial_z^2). \end{aligned}$$

Let $L_\epsilon(\eta)$ be an operator on $L_\alpha^2(\mathbb{R})$ with $D(L_\epsilon) = H_\alpha^3(\mathbb{R})$ for an $\hat{\alpha} \in (0, \hat{\alpha}_\epsilon)$ and

$$(\partial_z^{-1}f)(z) = -\int_z^\infty f(z_1) dz_1 \quad \text{for } f \in L_\alpha^2(\mathbb{R}).$$

We remark that $F(U, \Lambda, 0, \eta) = 2\mathcal{L}_{KP}(\eta)U - 2\Lambda U$ and the translated eigenvalue problem (3.12) is similar to the eigenvalue problem of the KP-II equation provided ϵ is sufficiently small. For small $\eta \neq 0$, (3.9) has two eigenvalues in the vicinity of 0.

First, we will find an approximate solution of (3.12). Let $U(\eta) = U_0 + \eta U_1 + \eta^2 U_2 + O(\eta^3)$, $\Lambda(\eta) = i\Lambda_{1,\epsilon}^0 \eta - \Lambda_{2,\epsilon}^0 \eta^2 + O(\eta^3)$ and formally equate the powers of η in (3.12). Then

$$(3.13) \quad L_\epsilon(0)U_0 = 0,$$

$$(3.14) \quad L_\epsilon(0)U_1 = \frac{i}{2}\Lambda_{1,\epsilon}^0 T_1(\epsilon, 0)U_0,$$

$$(3.15) \quad 2L_\epsilon(0)U_2 = -\{T_2(\epsilon, 0) + \Lambda_{2,\epsilon}^0 T_1(\epsilon, 0) - \epsilon^2(\Lambda_{1,\epsilon}^0)^2 B_\epsilon(0)\partial_z^{-1}\}U_0 + i\Lambda_{1,\epsilon}^0 T_1(\epsilon, 0)U_1.$$

Let $\theta_{1,\epsilon}(z) = \partial_c q_c(\frac{z}{\epsilon})$, $\theta_{\epsilon,d}(z) = d\theta_\epsilon(\sqrt{d}z)$ and $\tilde{\theta}_{1,\epsilon} = 2\partial_d \theta_{\epsilon,d}|_{d=1}$. By (1.4),

$$(3.16) \quad (bc^2 - a)\theta_\epsilon'' - \theta_\epsilon + \frac{3c}{2}\theta_\epsilon^2 = 0.$$

It follows from [35, Proposition 2.8] that

$$(3.17) \quad L_\epsilon(0)\theta'_\epsilon = 0, \quad L_\epsilon(0)\tilde{\theta}_{1,\epsilon} = -\theta'_\epsilon,$$

$$(3.18) \quad L_\epsilon(0)^*\theta_\epsilon = 0, \quad L_\epsilon(0)^*\int_{-\infty}^z \tilde{\theta}_{1,\epsilon}(z_1) dz_1 = \theta_\epsilon,$$

$$(3.19) \quad \ker_g(L_\epsilon(0)) = \text{span}\{\theta'_\epsilon, \tilde{\theta}_{1,\epsilon}\}, \quad \ker_g(L_\epsilon(0)^*) = \text{span}\left\{\theta_\epsilon, \int_{-\infty}^z \tilde{\theta}_{1,\epsilon}\right\},$$

where $\ker_g(A)$ denotes the generalized kernel of the operator A . Differentiating (1.4) with respect to c and x , using the change of variables (2.9), (2.10) and dropping the hats in the resulting equation, we have

$$(3.20) \quad L_\epsilon(0)\theta_{1,\epsilon} = -\frac{1}{2}T_1(\epsilon, 0)\theta'_\epsilon, \quad L_\epsilon(0)^*\int_{-\infty}^z \theta_{1,\epsilon} = \frac{1}{2}\partial_z^{-1}T_1(\epsilon, 0)\theta'_\epsilon.$$

Combining (3.13), (3.14), (3.17), (3.20) and the fact that $\ker(L_\epsilon(0)) = \text{span}\{\theta'_\epsilon\}$, we have

$$(3.21) \quad U_0 = \theta'_\epsilon, \quad U_1 = -i\Lambda_{1,\epsilon}^0\theta_{1,\epsilon} + C_1\theta'_\epsilon$$

up to the constant multiplicity, where C_1 is an arbitrary constant.

Next, we will determine $\Lambda_{1,\epsilon}^0$. Multiplying (3.15) by θ_ϵ and substituting (3.21) into the resulting equation, we have from (3.18)

$$\begin{aligned} & \langle T_2(\epsilon, 0)\theta'_\epsilon + \Lambda_{2,\epsilon}^0 T_1(\epsilon, 0)\theta'_\epsilon - \epsilon^2(\Lambda_{1,\epsilon}^0)^2 B_\epsilon(0)\theta_\epsilon, \theta_\epsilon \rangle + i\Lambda_{1,\epsilon}^0 \langle T_1(\epsilon, 0)(i\Lambda_{1,\epsilon}^0\theta_{1,\epsilon} - C_1\theta'_\epsilon), \theta_\epsilon \rangle \\ & = -2\langle U_2, L_\epsilon(0)^*\theta_\epsilon \rangle = 0. \end{aligned}$$

Since θ_ϵ is even and θ'_ϵ and $T_1(\epsilon, 0)\theta'_\epsilon$ are odd, we have $\langle T_1(\epsilon, 0)\theta'_\epsilon, \theta_\epsilon \rangle = \langle \theta'_\epsilon, \theta_\epsilon \rangle = 0$ and

$$(3.22) \quad (\Lambda_{1,\epsilon}^0)^2 = \frac{f_1(\epsilon)}{f_2(\epsilon)},$$

$$f_1(\epsilon) = \langle T_2(\epsilon, 0)\theta'_\epsilon, \theta_\epsilon \rangle, \quad f_2(\epsilon) = \langle T_1(\epsilon, 0)\theta_{1,\epsilon} + \epsilon^2 B_\epsilon(0)\theta_\epsilon, \theta_\epsilon \rangle.$$

By (3.16) and the fact that $(T_1(\epsilon, 0)\partial_z)^*\theta_\epsilon = -T_1(\epsilon, 0)\theta'_\epsilon = -c^{-1}\partial_z\{(A_\epsilon(0) + c^2 B_\epsilon(0))\theta_\epsilon\}$, we have

$$f_1(\epsilon) = \frac{1+2c^2}{3}\langle \theta_\epsilon, \theta_\epsilon \rangle + \frac{\epsilon^2}{3}(4a - bc^2)\langle \theta'_\epsilon, \theta'_\epsilon \rangle,$$

$$f_2(\epsilon) = \frac{1}{c}\langle \{A_\epsilon(0) + c^2 B_\epsilon(0)\}\theta_\epsilon, \theta_{1,\epsilon} \rangle + \epsilon^2\langle B_\epsilon(0)\theta_\epsilon, \theta_\epsilon \rangle.$$

Since

$$(3.23) \quad \|\theta_\epsilon - \theta_0\|_{H_\alpha^k(\mathbb{R}) \cap H_{-\alpha}^k(\mathbb{R})} + \|\theta_{1,\epsilon} - 2\theta_0 - z\theta'_0\|_{H^k(\mathbb{R}) \cap H_{-\alpha}^k(\mathbb{R})} = O(\epsilon^2) \quad \text{for any } k \geq 0,$$

we have $\Lambda_{1,\epsilon}^0 = \pm \frac{1}{\sqrt{3}} + O(\epsilon^2)$.

Now we will use the Lyapunov Schmidt method to prove existence of solutions to (3.12) satisfying $(U(\eta), \Lambda(\eta)) \simeq (\theta'_\epsilon - i\eta\Lambda_{1,\epsilon}^0\theta_{1,\epsilon}, i\eta\Lambda_{1,\epsilon}^0)$.

Lemma 3.3. *Let $\hat{\alpha} \in (0, \hat{\alpha}_0/2)$. There exist positive constants ϵ_0 and η_0 such that (3.12) has a solution $(U_\epsilon(\eta), \Lambda_\epsilon(\eta))$ satisfying for any $\eta \in [-\eta_0, \eta_0]$ and $k \geq 0$,*

$$(3.24) \quad \sup_{\epsilon \in (0, \epsilon_0)} \|U_\epsilon(\eta) - \theta'_\epsilon + \Lambda_\epsilon(\eta)\theta_{1,\epsilon}\|_{H^k_{\hat{\alpha}}(\mathbb{R})} = O(\eta^2),$$

$$(3.25) \quad \sup_{\epsilon \in (0, \epsilon_0)} |\Lambda_\epsilon(\eta) - i\Lambda_{1,\epsilon}^0\eta + \Lambda_{2,\epsilon}^0\eta^2| = O(\eta^3),$$

where $\Lambda_{1,\epsilon}^0$ and $\Lambda_{2,\epsilon}^0$ are constants satisfying $\Lambda_{1,\epsilon}^0 = \frac{1}{\sqrt{3}} + O(\epsilon^2)$ and $\Lambda_{2,\epsilon}^0 = \frac{2}{3\hat{\alpha}_0} + O(\epsilon^2)$. Moreover,

$$(3.26) \quad \overline{U_\epsilon(\eta)} = U_\epsilon(-\eta), \quad \overline{\Lambda_\epsilon(\eta)} = \Lambda_\epsilon(-\eta) \quad \text{for } \eta \in [-\eta_0, \eta_0],$$

and the mapping $[-\eta_0, \eta_0] \ni \eta \mapsto (U_\epsilon(\eta), \Lambda_\epsilon(\eta)) \in H^k_{\hat{\alpha}}(\mathbb{R}) \times \mathbb{R}$ is smooth for any $k \geq 0$.

Proof. Let $\Lambda(\eta) = i\eta\Lambda_1(\eta)$ and

$$(3.27) \quad U(\eta) = \theta'_\epsilon - \{i\eta\Lambda_1(\eta) - \eta^2\gamma(\eta)\}\theta_{1,\epsilon} + \eta^2\tilde{U}(\eta), \quad \tilde{U}(\eta) \perp \theta_\epsilon, \quad \int_{-\infty}^z \tilde{\theta}_{1,\epsilon}(z_1) dz_1.$$

Then (3.12) is translated into

$$(3.28) \quad 2\tilde{L}_\epsilon(\eta)\tilde{U} + G_1(\gamma, \Lambda_1, \epsilon, \eta) - i\eta G_2(\gamma, \Lambda_1, \epsilon, \eta) = 0,$$

where

$$\begin{aligned} \tilde{L}_\epsilon(\eta) &= L_\epsilon(\eta) - \frac{i}{2}\eta\Lambda_1(\eta)T_1(\epsilon, \eta) - \frac{\epsilon^2}{2}\eta^2\Lambda_1(\eta)^2B_\epsilon(\eta)\partial_z^{-1}, \\ G_1(\gamma, \Lambda_1, \epsilon, \eta) &= T_2(\epsilon, \eta)\theta'_\epsilon + 2\gamma L_\epsilon(\eta)\theta_{1,\epsilon} - \Lambda_1^2\{T_1(\epsilon, \eta)\theta_{1,\epsilon} + \epsilon^2B_\epsilon(\eta)\theta_\epsilon\}, \\ G_2(\gamma, \Lambda_1, \epsilon, \eta) &= 2b\epsilon\epsilon^4\Lambda_1\theta'_\epsilon + \Lambda_1\{T_2(\epsilon, \eta) + \gamma T_1(\epsilon, \eta)\}\theta_{1,\epsilon} \\ &\quad + \epsilon^2\Lambda_1^2(\Lambda_1 + i\gamma\eta)B_\epsilon(\eta) \int_z^\infty \theta_{1,\epsilon}(z_1) dz_1. \end{aligned}$$

Here we use (3.20) and the fact that $\{T_1(\epsilon, \eta) - T_1(\epsilon, 0)\}\theta'_\epsilon = 2b\epsilon\epsilon^4\eta^2\theta'_\epsilon$.

Let $P_\epsilon : L^2_{\hat{\alpha}} \rightarrow \ker_g(L_\epsilon(0))$ be the spectral projection associated with $L_\epsilon(0)$ and let $Q_\epsilon = I - P_\epsilon(0)$. Since $\tilde{U} \in Q_\epsilon L^2_{\hat{\alpha}}(\mathbb{R})$, we can translate (3.28) into

$$(3.29) \quad 2\hat{L}_\epsilon(\eta)\tilde{U} + Q_\epsilon G_1(\gamma, \Lambda_1, \epsilon, \eta) - i\eta Q_\epsilon G_2(\gamma, \Lambda_1, \epsilon, \eta) = 0,$$

$$(3.30) \quad F_1(\gamma, \Lambda_1, \epsilon, \eta) := \left\langle G_1(\gamma, \Lambda_1, \epsilon, \eta) - i\eta G_2(\gamma, \Lambda_1, \epsilon, \eta) + 2\{\tilde{L}_\epsilon(\eta) - L_\epsilon(0)\}\tilde{U}, \theta_\epsilon \right\rangle,$$

$$(3.31) \quad F_2(\gamma, \Lambda_1, \epsilon, \eta) := \left\langle G_1(\gamma, \Lambda_1, \epsilon, \eta) - i\eta G_2(\gamma, \Lambda_1, \epsilon, \eta) + 2\{\tilde{L}_\epsilon(\eta) - L_\epsilon(0)\}\tilde{U}, \int_{-\infty}^z \tilde{\theta}_{1,\epsilon} \right\rangle,$$

where $\hat{L}_\epsilon(\eta) = Q_\epsilon \tilde{L}_\epsilon(\eta) Q_\epsilon$. Let k_1 be a positive number such that

$$\begin{aligned} \sup_{\epsilon \in (0, \epsilon_0], \eta \in [-\eta_0, \eta_0]} & \left(\|T_1(\epsilon, \eta)\|_{B(H^2_{\hat{\alpha}}(\mathbb{R}), L^2_{\hat{\alpha}}(\mathbb{R}))} + \|T_2(\epsilon, \eta)\|_{B(H^1_{\hat{\alpha}}(\mathbb{R}), L^2_{\hat{\alpha}}(\mathbb{R}))} \right. \\ & \left. + \|B_\epsilon(\eta)\partial_z^{-1}\|_{B(H^1_{\hat{\alpha}}(\mathbb{R}), L^2_{\hat{\alpha}}(\mathbb{R}))} \right) \leq k_1. \end{aligned}$$

Suppose $\sup_{\eta \in [-\eta_0, \eta_0]} (|\Lambda_1(\eta)| + |\gamma(\eta)|) \leq k_2$ for a $k_2 > 0$. Since $\|Q_\epsilon L_\epsilon(0)^{-1} Q_\epsilon\|_{B(L_\alpha^2(\mathbb{R}), H_\alpha^3(\mathbb{R}))}$ is uniformly bounded in $\epsilon \in (0, \epsilon_0)$ and

$$\|\widehat{L}_\epsilon(\eta) - Q_\epsilon L_\epsilon(0) Q_\epsilon\|_{B(H_\alpha^2(\mathbb{R}), L_\alpha^2(\mathbb{R}))} \lesssim \eta^2 k_1 (1 + k_2^2 \epsilon^2) + \eta k_1 k_2,$$

we see that $\widehat{L}_\epsilon(\eta)^{-1} : Q_\epsilon L_\alpha^2(\mathbb{R}) \rightarrow Q_\epsilon H_\alpha^3(\mathbb{R})$ is uniformly bounded in $\epsilon \in (0, \epsilon_0)$ and $\eta \in [-\eta_0, \eta_0]$ provided ϵ_0 and η_0 are sufficiently small. Thus there exists a positive constant C_1 such that

$$\sup_{\epsilon \in (0, \epsilon_0], \eta \in [-\eta_0, \eta_0]} \|\widetilde{U}(\eta)\|_{H_\alpha^3(\mathbb{R})} \leq C_1 \{(1 + k_2)^2 + \epsilon_0^2 \eta_0 k_2^3\}.$$

Let

$$(3.32) \quad \gamma_\epsilon^0 = \frac{f_3(\epsilon)}{f_4(\epsilon)},$$

$$f_3(\epsilon) = (\Lambda_{1,\epsilon}^0)^2 \left\langle T_1(\epsilon, 0) \theta_{1,\epsilon} + \epsilon^2 B_\epsilon(0) \theta_\epsilon, \int_{-\infty}^z \tilde{\theta}_{1,\epsilon}(z_1) dz_1 \right\rangle - \left\langle T_2(\epsilon, 0) \theta'_\epsilon, \int_{-\infty}^z \tilde{\theta}_{1,\epsilon}(z_1) dz_1 \right\rangle,$$

$$f_4(\epsilon) = 2 \left\langle L_\epsilon(0) \theta_{1,\epsilon}, \int_{-\infty}^z \tilde{\theta}_{1,\epsilon}(z_1) dz_1 \right\rangle.$$

By (3.18) and (3.23), $f_4(\epsilon) = 3\langle \theta_0, \theta_0 \rangle + O(\epsilon^2)$. Using (3.22), (3.23) and the fact that $(\Lambda_{1,\epsilon}^0)^2 = \frac{1}{3} + O(\epsilon^2)$ and

$$(3.33) \quad \|\tilde{\theta}_{1,\epsilon} - 2\theta_0 - z\theta'_0\|_{H_\alpha^k(\mathbb{R}) \cap H_{-\alpha}^k(\mathbb{R})} = O(\epsilon^2) \quad \text{for any } k \geq 0,$$

we have

$$f_3(\epsilon) = \left\langle \frac{2}{3} \theta_{1,\epsilon} - \theta_\epsilon, \int_{-\infty}^z \tilde{\theta}_{1,\epsilon} \right\rangle + O(\epsilon^2) = -\frac{1}{6} \|\theta_0\|_{L^1(\mathbb{R})}^2 + O(\epsilon^2).$$

Thus we have

$$\gamma_\epsilon^0 = -\frac{1}{18} \frac{\|\theta_0(z)\|_{L^1(\mathbb{R})}^2}{\langle \theta_0, \theta_0 \rangle} + O(\epsilon^2) = -\frac{1}{3\hat{\alpha}_0} + O(\epsilon^2).$$

In view of (3.22) and (3.32),

$$\begin{aligned} F_1(\widetilde{U}_0, \gamma_\epsilon^0, \Lambda_{1,\epsilon}^0, \epsilon, 0) &= \langle G_1(\gamma, \Lambda_{1,\epsilon}^0, \epsilon, 0), \theta_\epsilon \rangle \\ &= \langle T_2(\epsilon, 0) \theta'_\epsilon, \theta_\epsilon \rangle - (\Lambda_{1,\epsilon}^0)^2 \langle T_1(\epsilon, 0) \theta_{1,\epsilon} + \epsilon^2 B_\epsilon(0) \theta_\epsilon, \theta_\epsilon \rangle \\ &= 0, \end{aligned}$$

where $\widetilde{U}_0 = \widetilde{U}(0)$.

$$F_2(\widetilde{U}_0, \gamma_\epsilon^0, \Lambda_{1,\epsilon}^0, \epsilon, 0) = \left\langle G_1(\gamma_\epsilon^0, \Lambda_{1,\epsilon}^0, \epsilon, 0), \int_{-\infty}^z \tilde{\theta}_{1,\epsilon}(z_1) dz_1 \right\rangle = 0,$$

Next, we compute the Fréchet derivative of (F_1, F_2) at $\mathcal{U}_0 = (\tilde{U}_0, \gamma_\epsilon^0, \Lambda_{1,\epsilon}^0, \epsilon, 0)$. By (3.18), (3.20), (3.23) and (3.33),

$$\begin{aligned}\partial_\gamma F_1(\mathcal{U}_0) &= 2\langle L_\epsilon(0)\theta_{1,\epsilon}, \theta_\epsilon \rangle = 0, \\ \partial_{\Lambda_1} F_1(\mathcal{U}_0) &= -2\Lambda_{1,\epsilon}^0 \langle T_1(\epsilon, 0)\theta_{1,\epsilon} + \epsilon^2 B_\epsilon(0)\theta_\epsilon, \theta_\epsilon \rangle = -6\Lambda_{1,\epsilon}^0 \langle \theta_0, \theta_0 \rangle + O(\epsilon^2), \\ \partial_\gamma F_2(\mathcal{U}_0) &= 2\langle L_\epsilon(0)\theta_{1,\epsilon}, \int_{-\infty}^z \tilde{\theta}_{1,\epsilon} \rangle = 2\langle \theta_{1,\epsilon}, \theta_\epsilon \rangle = 3\langle \theta_0, \theta_0 \rangle + O(\epsilon^2), \\ \partial_{\Lambda_1} F_2(\mathcal{U}_0) &= -2\Lambda_{1,\epsilon}^0 \left\langle T_1(\epsilon, 0)\theta_{1,\epsilon} + \epsilon^2 B_\epsilon(0)\theta_\epsilon, \int_{-\infty}^z \tilde{\theta}_{1,\epsilon} \right\rangle = -2\Lambda_{1,\epsilon}^0 \|\theta_0\|_{L^1}^2 + O(\epsilon^2).\end{aligned}$$

and $D_{(\gamma, \Lambda_1)}(F_1, F_2)(\mathcal{U}_0) = \begin{pmatrix} \partial_\gamma F_1(\mathcal{U}_0) & \partial_{\Lambda_1} F_1(\mathcal{U}_0) \\ \partial_\gamma F_2(\mathcal{U}_0) & \partial_{\Lambda_1} F_2(\mathcal{U}_0) \end{pmatrix}$ is invertible. Thus by the implicit function theorem, there exists a smooth curve $(\gamma_\epsilon(\eta), \Lambda_{1,\epsilon}(\eta))$ around $\eta = 0$ satisfying

$$(3.34) \quad \gamma_\epsilon(0) = \gamma_\epsilon^0, \quad \Lambda_{1,\epsilon}(0) = \Lambda_{1,\epsilon}^0, \quad \Lambda'_{1,\epsilon}(0) = -\frac{\partial_\eta F_1(\mathcal{U}_0)}{\partial_{\Lambda_1} F_1(\mathcal{U}_0)} =: i\Lambda_{2,\epsilon}^0.$$

Since

$$\begin{aligned}G_2(\gamma_\epsilon(0), \Lambda_{1,\epsilon}(0), \epsilon, 0) &= \Lambda_{1,\epsilon}(0) \{T_2(\epsilon, 0) + \gamma_\epsilon(0)T_1(\epsilon, 0)\}\theta_{1,\epsilon} + O(\epsilon^2) \\ &= \Lambda_{1,\epsilon}^0 \left\{ -\int_z^\infty \theta_{1,\epsilon}(z_1) dz_1 + 2\gamma_\epsilon(0)\theta_{1,\epsilon} \right\} + O(\epsilon^2) \quad \text{in } L_\alpha^2(\mathbb{R}),\end{aligned}$$

we have

$$\begin{aligned}\partial_\eta F_1(\tilde{U}_0, \gamma_\epsilon^0, \Lambda_{1,\epsilon}^0, \epsilon, 0) &= -i \left\langle G_2(\tilde{U}_0, \gamma_\epsilon^0, \Lambda_{1,\epsilon}^0, \epsilon, 0), \theta_\epsilon \right\rangle, \\ &= i\Lambda_{1,\epsilon}^0 \left\{ \frac{1}{2} \|\theta_0\|_{L^1(\mathbb{R})}^2 - 3\gamma_\epsilon^0 \langle \theta_0, \theta_0 \rangle \right\} + O(\epsilon^2) \\ &= \frac{2i}{3} \Lambda_{1,\epsilon}^0 \|\theta_0\|_{L^1(\mathbb{R})}^2,\end{aligned}$$

and $\Lambda_{2,\epsilon}^0 = \frac{1}{9} \|\theta_0\|_{L^1(\mathbb{R})}^2 \|\theta_0\|_{L^2(\mathbb{R})}^{-2} + O(\epsilon^2) = 2/(3\hat{\alpha}) + O(\epsilon^2)$.

Letting $\Lambda_\epsilon(\eta) = i\eta\Lambda_{1,\epsilon}(\eta)$ and

$$\begin{aligned}U_\epsilon(\eta) &= \theta'_\epsilon - \{\Lambda_\epsilon(\eta) - \eta^2 \gamma_\epsilon(\eta)\}\theta_{1,\epsilon} \\ &\quad - \frac{\eta^2}{2} \hat{L}_\epsilon(\eta)^{-1} Q_\epsilon \{G_1(\gamma_\epsilon(\eta), \Lambda_{1,\epsilon}(\eta), \epsilon, \eta) - i\eta G_2(\gamma_\epsilon(\eta), \Lambda_{1,\epsilon}(\eta), \epsilon, \eta)\},\end{aligned}$$

we have (3.24) and (3.26) because $\overline{\tilde{L}_\epsilon(\eta)} = \tilde{L}_\epsilon(-\eta)$ and $\overline{F_j(\gamma, \Lambda, \eta, \epsilon)} = F_j(\overline{\gamma}, \overline{\Lambda}, -\eta, \epsilon)$ for $j = 1, 2$. Thus we complete the proof. \square

Lemma 3.4. *Let $c, \hat{\alpha}, \epsilon_0$ and η_0 be as in Lemma 3.3. For any $\epsilon \in (0, \epsilon_0)$ and $\eta \in [-\epsilon^2 \eta_0, \epsilon^2 \eta_0]$, let $\lambda(\eta) = \epsilon^3 \Lambda_\epsilon(\epsilon^{-2} \eta)$, $u(z, \eta) = {}^t(u_1(z, \eta), u_2(z, \eta))$, $v(z, \eta) = {}^t(v_1(z, \eta), v_2(z, \eta))$ and*

$$\begin{aligned} u_1(z, \eta) &= \partial_z^{-1} U_\epsilon(\epsilon z, \epsilon^{-2} \eta), \\ u_2(z, \eta) &= -c\epsilon U_\epsilon(\epsilon z, \epsilon^{-2} \eta) + \lambda(\eta)(\partial_z^{-1} U_\epsilon)(\epsilon z, \epsilon^{-2} \eta), \\ v_1(z, \eta) &= (\lambda(-\eta) + c\partial_z)B(\eta) \int_{-\infty}^{\epsilon z} U_\epsilon(-z_1, -\epsilon^{-2} \eta) dz_1 \\ &\quad - (2q_c \partial_z + q'_c) \int_{-\infty}^{\epsilon z} U_\epsilon(-z_1, -\epsilon^{-2} \eta) dz_1, \\ v_2(z, \eta) &= B(\eta) \int_{-\infty}^{\epsilon z} U_\epsilon(-z_1, -\epsilon^{-2} \eta) dz_1. \end{aligned}$$

Then

$$(3.35) \quad \mathcal{L}(\eta)u(\cdot, \eta) = \lambda(\eta)u(\cdot, \eta), \quad \mathcal{L}(\eta)^*v(\cdot, \eta) = \lambda(-\eta)v(\cdot, \eta),$$

$$(3.36) \quad \overline{\lambda(\eta)} = \lambda(-\eta), \quad \overline{u(z, \eta)} = u(z, -\eta), \quad \overline{v(z, \eta)} = v(z, -\eta),$$

$$(3.37) \quad \langle u(x, \eta), v(x, -\eta) \rangle = 0 \quad \text{for } \eta \in [-\epsilon^2 \eta_0, \epsilon^2 \eta_0] \setminus \{0\}.$$

Moreover, for any $k \in \mathbb{N}$, the mappings $[-\epsilon^2 \eta_0, \epsilon^2 \eta_0] \ni \eta \mapsto u(\epsilon^{-1} \cdot, \eta) \in H_\alpha^k(\mathbb{R}) \times H_\alpha^{k-1}(\mathbb{R})$ and $[-\epsilon^2 \eta_0, \epsilon^2 \eta_0] \ni \eta \mapsto v(\epsilon^{-1} \cdot, \eta) \in H_{-\alpha}^k(\mathbb{R}) \times H_{-\alpha}^{k-1}(\mathbb{R})$ are smooth.

Proof. By (3.10), (3.11) and the definition of $U_\epsilon(\eta)$, we see that $u(z, \eta)$ is a solution of (3.9) with $\lambda = \lambda(\eta)$. The mappings $\eta \mapsto u(\epsilon^{-1} \cdot, \eta)$ and $v(\epsilon^{-1} \cdot, \eta)$ are smooth thanks to the smoothness of $U_\epsilon(\eta)$ and (3.36) follows from (3.26).

Suppose $\mathcal{L}(\eta)^* \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda(-\eta) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\tilde{v}_2 = B(\eta)^{-1} v_2$. Then

$$(3.38) \quad v_1 = (\lambda(-\eta) + c\partial_z)B(\eta)\tilde{v}_2 + v_{2,c}(\eta)^* \tilde{v}_2,$$

$$(3.39) \quad \{A(\eta)(\partial_z^2 - \eta^2) - (\lambda(-\eta) + c\partial_z)^2 B(\eta)\} \tilde{v}_2 - \{v_{1,c}(\eta)^* + (\lambda(-\eta) + c\partial_z)v_{2,c}(\eta)^*\} \tilde{v}_2 = 0.$$

Formally, we have $v_{2,c}(\eta)^* = -v_{2,c}(\eta)$ and $v_{1,c}(\eta)^* + c\partial_z v_{2,c}(\eta)^* = v_{1,c}(\eta) - cv_{2,c}(\eta)\partial_z$. Using the change of variable $z \mapsto -z$ and the fact that q_c is an even function, we see that $\tilde{v}_2(-z)$ satisfies (3.10) with $\lambda = \lambda(-\eta)$ and that

$$\tilde{v}_2(z, \eta) = \int_{-\infty}^{\epsilon z} U_\epsilon(-z_1, -\epsilon^{-2} \eta) dz_1$$

is a solution of (3.39). Thus we prove $\mathcal{L}(\eta)^*v(\cdot, \eta) = \lambda(-\eta)v(\cdot, \eta)$. We have (3.37) from (3.35) since $\overline{\lambda(\eta)} \neq \lambda(\eta)$ for $\eta \in [-\epsilon^2 \eta_0, \epsilon^2 \eta_0] \setminus \{0\}$. Thus we complete the proof. \square

Let

$$\begin{aligned} g(z, \eta) &= \frac{\sqrt{3}}{2} \left(1 + i \frac{\Re \langle u(\cdot, \eta), v(\cdot, \eta) \rangle}{\Im \langle u(\cdot, \eta), v(\cdot, \eta) \rangle} \right) \begin{pmatrix} u_1(z, \eta) \\ u_2(z, \eta) \end{pmatrix}, \\ g^*(z, \eta) &= -\frac{\hat{\alpha}_0}{4} \begin{pmatrix} v_1(z, \eta) \\ v_2(z, \eta) \end{pmatrix}, \end{aligned}$$

By (3.36) and (3.37),

$$(3.40) \quad \overline{g(z, \eta)} = g(z, -\eta), \quad \overline{g^*(z, \eta)} = g^*(z, -\eta),$$

$$(3.41) \quad \langle g(\cdot, \eta), g^*(\cdot, -\eta) \rangle = 0 \quad \text{and} \quad \Re \langle g(\cdot, \eta), g^*(\cdot, \eta) \rangle = 0 \quad \text{for } \eta \in [-\epsilon^2 \eta_0, \epsilon^2 \eta_0].$$

To resolve the degeneracy of the subspace $\text{span}\{g(\cdot, \eta), g(\cdot, -\eta)\}$ at $\eta = 0$, we introduce

$$(3.42) \quad g_1(z, \eta) = \frac{1}{2}\{g(z, \eta) + g(z, -\eta)\}, \quad g_2(z, \eta) = \frac{1}{2i\kappa(\eta)}\{g(z, \eta) - g(z, -\eta)\},$$

$$(3.43) \quad g_1^*(z, \eta) = \frac{i}{2\kappa(\eta)}\{g^*(z, \eta) - g^*(z, -\eta)\}, \quad g_2^*(z, \eta) = \frac{1}{2}\{g^*(z, \eta) + g^*(z, -\eta)\},$$

where $\kappa(\eta) = \frac{1}{2}\Im \langle g(\cdot, \eta), g^*(\cdot, \eta) \rangle$. By (3.40) and (3.41), we have

$$(3.44) \quad \langle g_i(\cdot, \eta), g_j^*(\cdot, \eta) \rangle = \delta_{ij} \quad \text{for } i, j = 1, 2.$$

The profiles of $g_k(z, \eta)$ and $g_k^*(z, \eta)$ for small line solitary waves are as follows.

Corollary 3.5. *Let c , $\hat{\alpha}$, ϵ_0 and η_0 be as in Lemma 3.3. For every $k \geq 0$, there exists a positive constant C such that for $\eta \in [-\epsilon^2 \eta_0, \epsilon^2 \eta_0]$ and $\epsilon \in (0, \epsilon_0]$,*

$$(3.45) \quad \left\| \begin{pmatrix} 1 & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \left\{ g_1(\epsilon^{-1} \cdot, \epsilon^2 \eta) - \frac{\sqrt{3}}{2} \begin{pmatrix} \theta_\epsilon \\ -\epsilon \theta'_\epsilon \end{pmatrix} \right\} \right\|_{H_{\hat{\alpha}}^k(\mathbb{R}) \times H_{\hat{\alpha}}^{k-1}(\mathbb{R})} \leq C(\epsilon^2 + \eta^2),$$

$$(3.46) \quad \left\| \begin{pmatrix} 1 & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \left\{ g_2(\epsilon^{-1} \cdot, \epsilon^2 \eta) - \frac{1}{2} \begin{pmatrix} \int_z^\infty \theta_{1,\epsilon} - 2\hat{\alpha}_0^{-1} \theta_\epsilon \\ \epsilon(\theta_{1,\epsilon} + 2\hat{\alpha}_0^{-1} \theta'_\epsilon) \end{pmatrix} \right\} \right\|_{H_{\hat{\alpha}}^k(\mathbb{R}) \times H_{\hat{\alpha}}^{k-1}(\mathbb{R})} \leq C(\epsilon^2 + \eta^2),$$

$$(3.47) \quad \left\| \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix} \left\{ g_1^*(\epsilon^{-1} \cdot, \epsilon^2 \eta) - \frac{\hat{\alpha}_0}{4\sqrt{3}} \begin{pmatrix} \epsilon \theta_{1,\epsilon} \\ \int_{-\infty}^z \theta_{1,\epsilon} \end{pmatrix} \right\} \right\|_{H_{-\hat{\alpha}}^k(\mathbb{R}) \times H_{-\hat{\alpha}}^{k-1}(\mathbb{R})} \leq C(\epsilon^2 + \eta^2),$$

$$(3.48) \quad \left\| \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix} \left\{ g_2^*(\epsilon^{-1} \cdot, \epsilon^2 \eta) - \frac{\hat{\alpha}_0}{4} \begin{pmatrix} \epsilon \theta'_\epsilon \\ \theta_\epsilon \end{pmatrix} \right\} \right\|_{H_{-\hat{\alpha}}^k(\mathbb{R}) \times H_{-\hat{\alpha}}^{k-1}(\mathbb{R})} \leq C(\epsilon^2 + \eta^2).$$

Proof. First, we expand $\langle u(\cdot, \epsilon^2 \eta), v(\cdot, \epsilon^2 \eta) \rangle$ into powers of η up to the second order. By the definitions of $u(z, \eta)$ and $v(z, \eta)$,

$$\begin{aligned} \langle u(\cdot, \epsilon^2 \eta), v(\cdot, \epsilon^2 \eta) \rangle &= 2\lambda(\epsilon^2 \eta) \langle u_1(\cdot, \epsilon^2 \eta), v_2(\cdot, \epsilon^2 \eta) \rangle - 2c \langle \partial_z u_1(\cdot, \epsilon^2 \eta), v_2(\cdot, \epsilon^2 \eta) \rangle \\ &\quad - \left\langle u_1(\cdot, \epsilon^2 \eta), 2\epsilon q_c U_\epsilon(-\epsilon \cdot, -\eta) + q'_c \int_{-\infty}^{\epsilon \cdot} U_\epsilon(-z_1, -\eta) dz_1 \right\rangle. \end{aligned}$$

By (3.27) and (3.34),

$$u_1(\epsilon^{-1} z, \epsilon^2 \eta) = \theta_\epsilon + \{i\eta \Lambda_{1,\epsilon}^0 - \eta^2(\gamma_\epsilon^0 + \Lambda_{2,\epsilon}^0)\} \int_z^\infty \theta_{1,\epsilon} - \eta^2 \int_z^\infty \tilde{U}_0(z_1) dz_1 + O(\eta^3) \quad \text{in } H_{\hat{\alpha}}^1(\mathbb{R}),$$

and

$$\begin{aligned} v_2(\epsilon^{-1} z, \epsilon^2 \eta) &= B_\epsilon(\eta) \left[-\theta_\epsilon + \{i\eta \Lambda_{1,\epsilon}^0 + \eta^2(\gamma_\epsilon^0 + \Lambda_{2,\epsilon}^0)\} \int_{-\infty}^z \theta_{1,\epsilon} \right] \\ &\quad + \eta^2 \int_{-\infty}^z B_\epsilon(0) \tilde{U}_0(-z_1) dz_1 + O(\eta^3) \quad \text{in } H_{-\hat{\alpha}}^1(\mathbb{R}). \end{aligned}$$

Using the fact that θ_ϵ and $\theta_{1,\epsilon}$ are even, we have

$$\begin{aligned} \epsilon \langle u_1(\cdot, \epsilon^2 \eta), v_2(\cdot, \epsilon^2 \eta) \rangle &= - \langle B_\epsilon(0) \theta_\epsilon, \theta_\epsilon \rangle - i \eta \Lambda_{1,\epsilon}^0 \left(\int_{\mathbb{R}} B_\epsilon(0) \theta_\epsilon \right) \left(\int_{\mathbb{R}} \theta_{1,\epsilon} \right) + O(\eta^2), \\ \langle \partial_z u_1(\cdot, \epsilon^2 \eta), v_2(\cdot, \epsilon^2 \eta) \rangle &= \{ 2i \eta \Lambda_{1,\epsilon}^0 - 2\eta^2 (\gamma_\epsilon^0 + \Lambda_{2,\epsilon}^0) \} \langle B_\epsilon(0) \theta_\epsilon, \theta_{1,\epsilon} \rangle \\ &\quad - \eta^2 \left\{ (\Lambda_{1,\epsilon}^0)^2 \langle B_\epsilon(0) \theta_{1,\epsilon}, \int_{-\infty}^z \theta_{1,\epsilon} \rangle + 2 \langle \tilde{U}_0, B_\epsilon(0) \theta_\epsilon \rangle \right\} + O(\eta^3), \\ \left\langle u_1(\cdot, \epsilon^2 \eta), 2\epsilon q_c U_\epsilon(-\epsilon \cdot, -\eta) + q'_c \int_{-\infty}^{\epsilon z} U_\epsilon(-z_1, -\eta) dz_1 \right\rangle &= -3i\epsilon^2 \eta \Lambda_{1,\epsilon}^0 \langle \theta_\epsilon^2, \theta_{1,\epsilon} \rangle + O(\epsilon^2 \eta^2). \end{aligned}$$

In the last line, we use (2.10). Since $\tilde{U}(\eta) \perp \theta_\epsilon$ and $\|B_\epsilon(0)\theta_\epsilon - \theta_\epsilon\|_{L_{-\hat{\alpha}}^2} = O(\epsilon^2)$, we have $\langle \tilde{U}_0, B_\epsilon(0)\theta_\epsilon \rangle = O(\epsilon^2)$. Combining the above with (3.23) and the fact that $\lambda(\epsilon^2 \eta) = \epsilon^3 \{ i \eta \Lambda_{1,\epsilon}^0 + O(\eta^2) \}$, we have

$$\begin{aligned} \Im \langle u(\cdot, \epsilon^2 \eta), v(\cdot, \epsilon^2 \eta) \rangle &= -2c \Im \langle \partial_z u_1(\cdot, \epsilon^2 \eta), v_2(\cdot, \epsilon^2 \eta) \rangle + O(\epsilon^2 \eta + \eta^3) \\ &= -4\eta \Lambda_{1,\epsilon}^0 \langle B_\epsilon(0) \theta_\epsilon, \theta_{1,\epsilon} \rangle + O(\epsilon^2 \eta + \eta^3) \\ &= \left\{ -\frac{16}{\sqrt{3}\hat{\alpha}_0} + O(\epsilon^2) \right\} \eta + O(\eta^3), \\ \Re \langle u(\cdot, \epsilon^2 \eta), v(\cdot, \epsilon^2 \eta) \rangle &= -2c \Re \langle \partial_z u_1(\cdot, \epsilon^2 \eta), v_2(\cdot, \epsilon^2 \eta) \rangle + O(\epsilon^2 \eta^2) \\ &= 2\eta^2 \left\{ (\Lambda_{1,\epsilon}^0)^2 \langle \theta_{1,\epsilon}, B_\epsilon(0) \int_{-\infty}^z \theta_{1,\epsilon} \rangle + 2(\gamma_\epsilon^0 + \Lambda_{2,\epsilon}^0) \langle B_\epsilon(0) \theta_\epsilon, \theta_{1,\epsilon} \rangle + O(\epsilon^2 + \eta^2) \right\} \\ &= \frac{32}{3\hat{\alpha}_0^2} \eta^2 + O(\epsilon^2 \eta^2 + \eta^4). \end{aligned}$$

Note that $\Re \langle u(\cdot, \epsilon^2 \eta), v(\cdot, \epsilon^2 \eta) \rangle$ is even in η thanks to (3.36). Thus we have

$$\begin{aligned} \frac{\Re \langle u(\cdot, \epsilon^2 \eta), v(\cdot, \epsilon^2 \eta) \rangle}{\Im \langle u(\cdot, \epsilon^2 \eta), v(\cdot, \epsilon^2 \eta) \rangle} &= -\frac{2\eta}{\sqrt{3}\hat{\alpha}_0} + O(\epsilon^2 \eta), \\ \langle g(\cdot, \epsilon^2 \eta), g^*(\cdot, \epsilon^2 \eta) \rangle &= -\frac{\sqrt{3}\hat{\alpha}_0}{8} i \Im \langle u(\cdot, \epsilon^2 \eta), v(\cdot, \epsilon^2 \eta) \rangle \left\{ 1 + \left(\frac{\Re \langle u(\cdot, \epsilon^2 \eta), v(\cdot, \epsilon^2 \eta) \rangle}{\Im \langle u(\cdot, \epsilon^2 \eta), v(\cdot, \epsilon^2 \eta) \rangle} \right)^2 \right\} \\ &= 2i\eta \{ 1 + O(\epsilon^2 + \eta^2) \}, \end{aligned}$$

and (3.45)–(3.48) follow immediately from the definitions of g_k and g_k^* ($k = 1, 2$). \square

Remark 3.1. In view of (3.40), we have

$$\begin{aligned} \mathcal{L}(\eta) g_1(\cdot, \eta) &= \Re \lambda(\eta) g_1(\cdot, \eta) - \kappa(\eta) \Im \lambda(\eta) g_2(\cdot, \eta), \\ \mathcal{L}(\eta) g_2(\cdot, \eta) &= \frac{\Im \lambda(\eta)}{\kappa(\eta)} g_1(\cdot, \eta) + \Re \lambda(\eta) g_2(\cdot, \eta), \\ \mathcal{L}(\eta)^* g_1^*(\cdot, \eta) &= \Re \lambda(\eta) g_1^*(\cdot, \eta) + \frac{\Im \lambda(\eta)}{\kappa(\eta)} g_2^*(\cdot, \eta), \\ \mathcal{L}(\eta)^* g_2^*(\cdot, \eta) &= -\kappa(\eta) \Im \lambda(\eta) g_1^*(\cdot, \eta) + \Re \lambda(\eta) g_2^*(\cdot, \eta). \end{aligned}$$

Now we define a spectral projection to resonant modes. Let $\mathcal{P}(\eta_0)$ be an operator defined by

$$\mathcal{P}(\eta_0)f(z, y) = \frac{1}{\sqrt{2\pi}} \sum_{k=1,2} \int_{-\eta_0}^{\eta_0} c_k(\eta) g_k(z, \eta) e^{iy\eta} d\eta,$$

$$c_k(\eta) = \int_{\mathbb{R}} (\mathcal{F}_y f)(z, \eta) \cdot g_k^*(z, \eta) dz$$

for $f \in X$ and let $\mathcal{Q}(\eta_0) = I - \mathcal{P}(\eta_0)$. Using Corollary 3.5, we can prove that $\mathcal{P}(\eta_0)$ and $\mathcal{Q}(\eta_0)$ are spectral projections associated with \mathcal{L} in exactly the same way with [28, Lemma 3.1].

Lemma 3.6. *Let $c = \sqrt{1 + \epsilon^2}$ and $\alpha \in (0, \hat{\alpha}_0/2)$. Then there exist positive constants ϵ_0 and η_1 such that for any $\epsilon \in (0, \epsilon_0)$ and $\eta_0 \in [0, \eta_1]$,*

- (1) $\|\mathcal{P}(\epsilon^2 \eta_0)f\|_X \leq C\|f\|_X$ for any $f \in X$, where C is a positive constant depending only on α, ϵ and η_1 ,
- (2) $\mathcal{L}\mathcal{P}(\epsilon^2 \eta_0)f = \mathcal{P}(\epsilon^2 \eta_0)\mathcal{L}f$ for any $f \in D(\mathcal{L})$,
- (3) $\mathcal{P}(\epsilon^2 \eta_0)^2 = \mathcal{P}(\epsilon^2 \eta_0)$ on X ,
- (4) $e^{t\mathcal{L}}\mathcal{P}(\epsilon^2 \eta_0) = \mathcal{P}(\epsilon^2 \eta_0)e^{t\mathcal{L}}$ on X .

4. PROPERTIES OF THE FREE OPERATOR \mathcal{L}_0

In this section, we investigate properties of the linearized operator \mathcal{L}_0 in X . To begin with, we investigate the spectrum of \mathcal{L}_0 .

Lemma 4.1. *Let $\alpha'_c = \sqrt{\frac{c-1}{bc-a}}$. Suppose $0 < a < b, c > 1$ and $\alpha \in (0, \alpha'_c)$. Then*

$$\sigma(\mathcal{L}_0(D)) \subset \left\{ \lambda \in \mathbb{C} \mid \Re \lambda < -\frac{\alpha}{2}(c-1) \right\}.$$

By (3.5), the operator $\begin{pmatrix} m_{11}(D) & m_{12}(D) \\ m_{21}(D) & m_{22}(D) \end{pmatrix}$ is bounded on X if and only if

$$(4.1) \quad \sum_{i,j=1,2} (1 + \xi^2 + \eta^2)^{(j-i)/2} |m_{ij}(\xi + i\alpha, \eta)| < \infty.$$

The symbol of the operator \mathcal{L}_0 is

$$\mathcal{L}_0(\xi, \eta) = \begin{pmatrix} ic\xi & 1 \\ -(\xi^2 + \eta^2)S(\xi, \eta)^2 & ic\xi \end{pmatrix}, \quad S(\xi, \eta) = \sqrt{\frac{1 + a(\xi^2 + \eta^2)}{1 + b(\xi^2 + \eta^2)}},$$

and we observe $L_0(\xi, \eta)P(\xi, \eta) = \text{diag}(\lambda_+(\xi, \eta), \lambda_-(\xi, \eta))P(\xi, \eta)$, where

$$(4.2) \quad \lambda_{\pm}(\xi, \eta) = ic\xi \pm i\mu(\xi, \eta)S(\xi, \eta), \quad \mu(\xi, \eta) = \xi\sqrt{1 + \xi^{-2}\eta^2},$$

$$P(\xi, \eta) = \begin{pmatrix} -i\mu(\xi, \eta)^{-1} & i\mu(\xi, \eta)^{-1} \\ S(\xi, \eta) & S(\xi, \eta) \end{pmatrix}.$$

To investigate properties of the resolvent operator $(\lambda - \mathcal{L}_0)^{-1}$, we need the following.

Claim 4.2. Suppose $0 < a < b$ and $\alpha > 0$. Then

$$(4.3) \quad 0 \leq \Im \mu(\xi + i\alpha, \eta) \leq \Im \mu(\xi + i\alpha, 0) = \alpha \quad \text{for } \xi \in \mathbb{R},$$

$$(4.4) \quad \xi \Re \mu(\xi + i\alpha, \eta) > 0, \quad \Im \mu(\xi + i\alpha, \eta) > 0 \quad \text{for } \xi \neq 0.$$

Claim 4.3. Suppose $0 < a < b$ and $0 < \alpha < \alpha_c$. Then

$$(4.5) \quad \Re S(\xi + i\alpha, \eta) > 0 \quad \text{for } (\xi, \eta) \in \mathbb{R}^2, ,$$

$$(4.6) \quad \xi \Im S(\xi + i\alpha, \eta) < 0 \quad \text{for } \xi \in \mathbb{R} \setminus \{0\} \text{ and } \eta \in \mathbb{R},$$

$$(4.7) \quad \sqrt{\frac{a}{b}} < |S(\xi + i\alpha, \eta)| < S(i\alpha, 0) < c \quad \text{for } (\xi, \eta) \in \mathbb{R}^2 \setminus \{(0, 0)\},$$

$$(4.8) \quad |S(\xi + i\alpha, \eta)| < 1 - \frac{b-a}{2} \frac{\xi^2 + \eta^2 - \alpha^2}{1 + b(\xi^2 + \eta^2 - \alpha^2)} \quad \text{for } (\xi, \eta) \in \mathbb{R}^2.$$

Claim 4.4. Suppose $0 < a < b$, $c > 1$ and $\alpha \in (0, \alpha'_c)$. Then for $(\xi, \eta) \in \mathbb{R}^2$,

$$(4.9) \quad -2\alpha c < \Re \lambda_+(\xi + i\alpha, \eta) \leq -\alpha c,$$

$$(4.10) \quad \Re \lambda_-(\xi + i\alpha, \eta) \leq -\alpha \left\{ c - 1 + \frac{b-a}{2} \frac{\xi^2 + \eta^2 - \alpha^2}{1 + b(\xi^2 + \eta^2 - \alpha^2)} \right\},$$

$$(4.11) \quad -\alpha c \leq \Re \lambda_-(\xi + i\alpha, \eta) \leq -\frac{\alpha}{2}(c-1).$$

Proof of Claim 4.2. Since

$$\mu(\xi + i\alpha, \eta) = (\xi + i\alpha) \sqrt{1 + \frac{\eta^2}{(\xi + i\alpha)^2}} = \operatorname{sgn}(\xi) \sqrt{(\xi + i\alpha)^2 + \eta^2},$$

we have (4.4).

Since $\Im \mu(i\alpha, \eta) = \sqrt{\alpha^2 - \eta^2}$ for $\eta \in [-\alpha, \alpha]$ and $\Im \mu(i\alpha, \eta) = 0$ for $\eta \in \mathbb{R}$ satisfying $|\eta| > \alpha$, we have (4.3) for $\xi = 0$. Let $s = \eta^2$, $\gamma_1(\xi, s) = \Re \mu(\xi + i\alpha, \eta)$ and $\gamma_2(\xi, s) = \Im \mu(\xi + i\alpha, \eta)$. To prove (4.3), it suffices to show that $\gamma_2(\xi, s)$ is monotone decreasing in s when $\xi \neq 0$. Differentiating

$$(4.12) \quad \gamma_1^2 - \gamma_2^2 = \xi^2 - \alpha^2 + s \quad \text{and} \quad \gamma_1 \gamma_2 = \alpha \xi$$

with respect to s , we have

$$(4.13) \quad \partial_s \gamma_2 = -\frac{\gamma_2}{2(\gamma_1^2 + \gamma_2^2)}.$$

Combining (4.13) with (4.4), we have $\partial_s \gamma_2 < 0$. Thus we prove (4.3). □

Proof of Claim 4.3. We observe

$$(4.14) \quad \begin{aligned} S(\xi + i\alpha, \eta)^2 &= \frac{1 + a(\xi^2 + \eta^2 - \alpha^2) + 2ia\alpha\xi}{1 + b(\xi^2 + \eta^2 - \alpha^2) + 2ib\alpha\xi} \\ &= \frac{a}{b} + \left(1 - \frac{a}{b}\right) \frac{1}{1 - b\alpha^2 + b(\xi^2 + \eta^2) + 2ib\alpha\xi}. \end{aligned}$$

Since $0 < a < b$ and $1 - b\alpha^2 > 0$ for $\alpha \in (0, \alpha_c)$, it follows from (4.14) that

$$(4.15) \quad |S(\xi + i\alpha, \eta)|^2 \geq \Re S(\xi + i\alpha, \eta)^2 > \frac{a}{b} > 0 \quad \text{for } (\xi, \eta) \in \mathbb{R}^2,$$

$$(4.16) \quad \xi \Im S(\xi + i\alpha, \eta)^2 < 0 \quad \text{for } \xi \in \mathbb{R} \setminus \{0\} \text{ and } \eta \in \mathbb{R}.$$

By (4.15), we have the first part of (4.7) and (4.5) because $\Re S(i\alpha, 0) = \sqrt{\frac{1-a\alpha^2}{1-b\alpha^2}} > 0$ and $S(\xi + i\alpha, \eta)$ is continuous in $(\xi, \eta) \in \mathbb{R}^2$. Eq. (4.6) follows from (4.5) and (4.16).

We have $c > S(i\alpha, 0)$ for $\alpha \in (0, \alpha_c)$. By (4.14) and the triangle inequality,

$$(4.17) \quad \begin{aligned} |S(\xi + i\alpha, \eta)|^2 &\leq \frac{a}{b} + \left(1 - \frac{a}{b}\right) \frac{1}{1 + b(\xi^2 + \eta^2 - \alpha^2)} \\ &= \frac{1 + a(\xi^2 + \eta^2 - \alpha^2)}{1 + b(\xi^2 + \eta^2 - \alpha^2)} \leq \frac{1 - a\alpha^2}{1 - b\alpha^2} = S(i\alpha, 0)^2, \end{aligned}$$

and $|S(\xi + i\alpha, \eta)| = S(i\alpha, 0)$ if and only if $\xi = \eta = 0$. Thus we have the second part of (4.7). Furthermore, we have (4.8) from (4.17) since $|S| \leq (|S|^2 + 1)/2$. Thus we complete the proof. \square

Using Claim 4.3, we will estimate the upper and lower bounds of $\lambda_{\pm}(\xi + i\alpha, \eta)$.

Proof of Claim 4.4. First, we will show

$$(4.18) \quad \Im(\mu(\xi + i\alpha, \eta)S(\xi + i\alpha, \eta)) \geq 0 \quad \text{for } (\xi, \eta) \in \mathbb{R}^2.$$

We see that $\mu(\xi + i\alpha, \eta)S(\xi + i\alpha, \eta)$ is a real number if and only if $\xi = 0$ and $|\eta| \geq \alpha$ since

$$\Im \{\mu(\xi + i\alpha, \eta)S(\xi + i\alpha, \eta)\}^2 = 2\alpha\xi \left[\frac{a}{b} + \left(1 - \frac{a}{b}\right) \frac{1}{|1 + b(\xi + i\alpha)^2 + b\eta^2|^2} \right]$$

and

$$\mu(i\alpha, \eta)^2 S(i\alpha, \eta)^2 = (\eta^2 - \alpha^2) \frac{1 - b\alpha^2 + \alpha\eta^2}{1 - b\alpha^2 + b\eta^2}.$$

Thanks to the continuity of $\Im(\mu S)(\xi + i\alpha, \eta)$ on \mathbb{R}^2 and the fact that $\Im(\mu S)(i\alpha, 0) > 0$, we have (4.18).

By (4.18) and the definition of λ_{\pm} ,

$$\Re \lambda_+(\xi + i\alpha, \eta) \leq -\alpha c, \quad \Re \lambda_-(\xi + i\alpha, \eta) \geq -\alpha c.$$

Since $0 < \alpha < \alpha'_c < \alpha_c$, it follows from (4.3), (4.4), (4.6) and (4.7) that

$$\begin{aligned} \Re \lambda_+(\xi + i\alpha, \eta) &\geq -\alpha c - \Im \mu(\xi + i\alpha, \eta) \Re S(\xi + i\alpha, \eta) \\ &> -2\alpha c, \end{aligned}$$

$$(4.19) \quad \Re \lambda_-(\xi + i\alpha, \eta) \leq -\alpha c + \Im \mu(\xi + i\alpha, \eta) \Re S(\xi + i\alpha, \eta).$$

Combining (4.19) with (4.3) and (4.8), we have (4.10). Since $x/(1+bx)$ is increasing on $[-\alpha^2, \infty)$ and $c > S(i\alpha, 0)^2$ for $\alpha \in (0, \alpha'_c)$,

$$\begin{aligned} \Re \lambda_-(\xi + i\alpha, \eta) &\leq -\alpha \left\{ c - 1 - \frac{b-a}{2} \frac{\alpha^2}{1-b\alpha^2} \right\} \\ &= -\alpha \left(c - \frac{1}{2} - \frac{1}{2} S(i\alpha)^2 \right) < -\frac{c-1}{2}. \end{aligned}$$

Thus we complete the proof. \square

Now we are in position to prove Lemma 4.1.

Proof of Lemma 4.1. If $\lambda \neq \lambda_{\pm}(\xi, \eta)$,

$$(4.20) \quad (\lambda - \mathcal{L}_0(\xi, \eta))^{-1} = \frac{1}{(\lambda - \lambda_+(\xi, \eta))(\lambda - \lambda_-(\xi, \eta))} \begin{pmatrix} \lambda - ic\xi & 1 \\ -(\mu S)^2(\xi, \eta) & \lambda - ic\xi \end{pmatrix}.$$

Since $2ic\xi = \lambda_+ + \lambda_-$ and $2i\mu S = \lambda_+ - \lambda_-$,

$$(4.21) \quad \begin{aligned} \frac{2(\lambda - ci\xi)}{(\lambda - \lambda_+(\xi, \eta))(\lambda - \lambda_-(\xi, \eta))} &= \frac{1}{\lambda - \lambda_+(\xi, \eta)} + \frac{1}{\lambda - \lambda_-(\xi, \eta)}, \\ \frac{2i\mu(\xi, \eta)S(\xi, \eta)}{(\lambda - \lambda_+(\xi, \eta))(\lambda - \lambda_-(\xi, \eta))} &= \frac{1}{\lambda - \lambda_+(\xi, \eta)} - \frac{1}{\lambda - \lambda_-(\xi, \eta)}. \end{aligned}$$

In view of (4.1), (4.7), (4.20) and (4.21), the operator $\lambda - \mathcal{L}_0$ has a bounded inverse on X if

$$(4.22) \quad \sup_{(\xi, \eta) \in \mathbb{R}^2} |\lambda - \lambda_{\pm}(\xi + i\alpha, \eta)|^{-1} < \infty.$$

Thus we have

$$(4.23) \quad \sigma(\mathcal{L}_0(D)) = \overline{\{\lambda_{\pm}(\xi + i\alpha, \eta) \mid (\xi, \eta) \in \mathbb{R}^2\}},$$

and Lemma 4.1 follows immediately from (4.9), (4.11) and (4.23). \square

To prove the boundedness of $(\lambda - \mathcal{L})^{-1}$ restricted on $\mathcal{Q}(\eta_0)X$ for a small $\eta_0 > 0$, the estimate (4.11) in Claim 4.4 is insufficient. To have a better estimate on $(\lambda - \lambda_-(D))^{-1}$, we will estimate $\lambda_-(\xi, \eta)$ in the high frequency regime, the middle frequency regime and in the low frequency regime, separately. Let $\delta = \epsilon^{1/20}$, $K = \delta^{-3}$ and

$$\begin{aligned} A_{high} &= \{(\xi, \eta) \in \mathbb{R}^2 \mid |\xi| \geq \delta \text{ or } |\eta| \geq \delta|\xi + i\alpha|\}, \\ A_{\xi, m} &= \{(\xi, \eta) \in \mathbb{R}^2 \mid K\epsilon \leq |\xi| \leq \delta, |\eta| \leq \delta|\xi + i\alpha|\}, \\ A_{\eta, m} &= \{(\xi, \eta) \in \mathbb{R}^2 \mid |\xi| \leq K\epsilon, K\epsilon|\xi + i\alpha| \leq |\eta| \leq \delta|\xi + i\alpha|\}, \\ A_{low} &= \{(\xi, \eta) \in \mathbb{R}^2 \mid |\xi| \leq K\epsilon, |\eta| \leq K\epsilon|\xi + i\alpha|\}, \\ \tilde{A}_{low} &= \{(\xi, \eta) \in \mathbb{R}^2 \mid |\xi| \leq K\epsilon, |\eta| \leq K(K + \hat{\alpha})\epsilon^2\}. \end{aligned}$$

Obviously, we have $\mathbb{R}^2 = A_{high} \cup A_{\xi, m} \cup A_{\eta, m} \cup A_{low}$ and $A_{low} \subset \tilde{A}_{low}$. Suppose $c = \sqrt{1 + \epsilon^2}$ and that ϵ is a small positive number. In the low frequency regime A_{low} ,

$$i\mu(D) \sim \epsilon \partial_{\hat{z}} + \frac{\epsilon^3}{2} \partial_{\hat{z}}^{-1} \partial_{\hat{y}}^2, \quad S(D) \sim I + \frac{b-a}{2} \partial_{\hat{z}}^2, \quad \lambda_-(D) \sim \epsilon^3 \mathcal{L}_{KP,0}(D_{\hat{z}}, D_{\hat{y}}), \quad \lambda_+(D) \sim 2\epsilon \partial_{\hat{z}},$$

where $\hat{z} = \epsilon z$, $\hat{y} = \epsilon^2 y$ and $\mathcal{L}_{KP,0}(D_{\hat{z}}, D_{\hat{y}}) = -\frac{1}{2}\{(b-a)\partial_{\hat{z}}^3 - \partial_{\hat{z}} + \partial_{\hat{z}}^{-1}\partial_{\hat{y}}^2\}$. More precisely, we have the following.

Lemma 4.5. *Let $c = \sqrt{1 + \epsilon^2}$, $\alpha = \epsilon\hat{\alpha}$ and $\hat{\alpha}_\epsilon = 1/\sqrt{bc^2 - a}$. Let $\xi = \epsilon\hat{\xi}$, $\eta = \epsilon^2\hat{\eta}$. Suppose $\hat{\alpha} \in (0, \hat{\alpha}_\epsilon)$. Then there exist positive constants ϵ_0 and C such that for $\epsilon \in (0, \epsilon_0)$,*

$$(4.24) \quad \lambda_-(\xi + i\alpha, \eta) = \frac{i\epsilon^3}{2}(\hat{\xi} + i\hat{\alpha}) \left\{ 1 + (b-a)(\hat{\xi} + i\hat{\alpha})^2 - \frac{\hat{\eta}^2}{(\hat{\xi} + i\hat{\alpha})^2} + O(K^4\epsilon^2) \right\}$$

for $(\xi, \eta) \in A_{low}$,

$$(4.25) \quad \lambda_-(\xi + i\alpha, \eta) = \frac{i\epsilon^3}{2}(\hat{\xi} + i\hat{\alpha}) \left\{ 1 + (b-a)(\hat{\xi} + i\hat{\alpha})^2 - \frac{\hat{\eta}^2}{(\hat{\xi} + i\hat{\alpha})^2} + O(K^8\epsilon^2) \right\}$$

for $(\xi, \eta) \in \tilde{A}_{low}$,

$$(4.26) \quad \Re\lambda_-(\xi + i\alpha, \eta) \leq -\frac{\hat{\alpha}\epsilon^3}{4} \left\{ 1 + (b-a)\hat{\xi}^2 \right\} \quad \text{for } (\xi, \eta) \in A_{\xi,m},$$

$$(4.27) \quad \Re\lambda_-(\xi + i\alpha, \eta) \leq -\frac{\alpha\epsilon^3}{4} \frac{\hat{\eta}^2}{\hat{\xi}^2 + \hat{\alpha}^2} \quad \text{for } (\xi, \eta) \in A_{\eta,m},$$

$$(4.28) \quad \Re\lambda_-(\xi + i\alpha, \eta) \leq -C\delta^2\epsilon \quad \text{for } (\xi, \eta) \in A_{high}.$$

Proof of Lemma 4.5. If $(\xi, \eta) \in A_{low}$, then

$$(4.29) \quad |\hat{\xi}| \leq K, \quad |\hat{\eta}|/|\hat{\xi} + i\hat{\alpha}| \leq K,$$

$$(4.30) \quad \begin{aligned} \mu(\xi + i\alpha, \eta) &= \epsilon(\hat{\xi} + i\hat{\alpha}) \sqrt{1 + \frac{\epsilon^2\hat{\eta}^2}{(\hat{\xi} + i\hat{\alpha})^2}} \\ &= \epsilon(\hat{\xi} + i\hat{\alpha}) \left\{ 1 + \frac{\epsilon^2}{2} \frac{\hat{\eta}^2}{(\hat{\xi} + i\hat{\alpha})^2} + O(K^4\epsilon^4) \right\}, \end{aligned}$$

$$(4.31) \quad \begin{aligned} S(\xi + i\alpha, \eta) &= \sqrt{1 + \frac{(a-b)\{(\xi + i\alpha)^2 + \eta^2\}}{1 + b\{(\xi + i\alpha)^2 + \eta^2\}}} \\ &= 1 + \frac{a-b}{2}\epsilon^2(\hat{\xi} + i\hat{\alpha})^2 + O(\epsilon^4K^4). \end{aligned}$$

Combining (4.29)–(4.31) and the fact that $c = 1 + \frac{\epsilon^2}{2} + O(\epsilon^4)$, we have (4.24). If $(\xi, \eta) \in \tilde{A}_{low}$, then $|\hat{\xi}| \leq K$ and $|\hat{\eta}|/|\hat{\xi} + i\hat{\alpha}| \leq K(K + \hat{\alpha})/\hat{\alpha}$ and we can prove (4.25) in exactly the same way.

Suppose $(\xi, \eta) \in A_{\xi,m}$. Then $\xi = O(\delta)$, $\alpha/\xi = O(K^{-1})$ and $\eta/\xi = O(\delta)$. By (4.10),

$$\begin{aligned} \Re\lambda_-(\xi + i\alpha, \eta) &\leq -\alpha \left\{ c - 1 + \frac{b-a}{2} \frac{\xi^2 + \eta^2 - \alpha^2}{1 + b(\xi^2 + \eta^2 - \alpha^2)} \right\} \\ &= -\alpha \left\{ \frac{\epsilon^2}{2} + O(\epsilon^4) + \frac{b-a}{2} (1 + O(\delta^2 + K^{-2})) \xi^2 \right\}. \end{aligned}$$

Thus we have (4.26) provided ϵ_0 , δ and K^{-1} are sufficiently small.

Let $(\xi, \eta) \in A_{\eta, m}$. By (4.3), (4.7) and (4.19),

$$(4.32) \quad \begin{aligned} \Re \lambda_-(\xi + i\alpha, \eta) &\leq -\alpha c + \Im \mu(\xi + i\alpha, \eta) \Re S(\xi + i\alpha, \eta) \\ &\leq -c \{ \alpha - \Im \mu(\xi + i\alpha, \eta) \}. \end{aligned}$$

Since

$$\begin{aligned} \mu(\xi + i\alpha, \eta) &= (\xi + i\alpha) \sqrt{1 + \frac{\eta^2}{(\xi + i\alpha)^2}} \\ &= \epsilon(\hat{\xi} + i\hat{\alpha}) \left\{ 1 + \frac{\epsilon^2 \hat{\eta}^2}{2(\hat{\xi} + i\hat{\alpha})^2} (1 + O(\delta^2)) \right\}, \end{aligned}$$

$$\Im \mu(\xi + i\alpha, \eta) = \epsilon \hat{\alpha} - \frac{\epsilon^3 \hat{\alpha} \hat{\eta}^2}{2(\hat{\xi}^2 + \hat{\alpha}^2)} (1 + O(\delta^2)).$$

By (4.32) and the above, we have (4.27) provided ϵ_0 , δ and K^{-1} are sufficiently small.

Finally, we will prove (4.28). Suppose $(\xi, \eta) \in A_{high}$ and $|\xi| \geq \delta$. Then there exists a positive constant C_1 such that $\xi^2 + \eta^2 - \alpha^2 \geq C_1 \delta^2$ and it follows from (4.10) that

$$\Re \lambda_-(\xi + i\alpha, \eta) \leq -\alpha \left\{ c - 1 + \frac{b-a}{2} \frac{C_1 \delta^2}{1 + b C_1 \delta^2} \right\} \lesssim -\epsilon \delta^2.$$

Suppose $(\xi, \eta) \in A_{high}$ and $|\eta| |\xi + i\alpha|^{-1} \geq \delta$. By (4.3) and (4.13),

$$(4.33) \quad \Im \mu(\xi + i\alpha, \eta) = \gamma_2(\xi, s) \leq \gamma_2(\xi, \delta^2 |\xi + i\alpha|^2) \quad \text{if } s = \eta^2 \geq \delta^2 |\xi + i\alpha|^2.$$

If $0 \leq s \leq \delta^2 |\xi + i\alpha|^2$,

$$\gamma_1^2 + \gamma_2^2 = |(\xi + i\alpha)^2 + s| \leq (1 + \delta^2) |\xi + i\alpha|^2,$$

and it follows from (4.13) that for a $C > 0$,

$$(4.34) \quad \gamma_2(\xi, \delta^2 |\xi + i\alpha|^2) \leq \gamma_2(\xi, 0) \exp(-\delta^2/2(1 + \delta^2)) \leq \alpha - C \delta^2.$$

Substituting (4.33) and (4.34) into (4.32), we have (4.28). Thus we complete the proof. \square

Finally, we will estimate operator norms of $(\lambda - \lambda_{\pm}(D))^{-1}$ on $L^2(\mathbb{R}_{\alpha}^2)$ and its subspaces. Let ρ_y and $\tilde{\rho}_y$ be functions on \mathbb{R} such that $\rho_y(\eta) + \tilde{\rho}_y(\eta) = 1$ for $\eta \in \mathbb{R}$ and

$$\rho_y(\eta) = \begin{cases} 1 & \text{if } |\eta| \leq K(K + \hat{\alpha})\epsilon^2, \\ 0 & \text{if } |\eta| \geq K(K + \hat{\alpha})\epsilon^2. \end{cases}$$

Let $\rho_z(\xi)$ be the characteristic function of $\{\xi \in \mathbb{C} \mid |\Re \xi| \leq K\epsilon\}$, $\tilde{\rho}_z(\xi) = 1 - \rho_z(\xi)$ and

$$Y := \rho_y(D_y) L_{\alpha}^2(\mathbb{R}^2), \quad Y_{low} := \rho_z(D_z) Y, \quad Y_{high} := \tilde{\rho}_z(D_z) Y.$$

We remark that $\tilde{A}_{low} = \text{supp } \rho_z(\xi) \rho_y(\eta)$.

Lemma 4.6. *Let c , α , and $\hat{\alpha}$ be as in Lemma 4.5. Let $\hat{\beta} \in (0, \frac{\hat{\alpha}}{8})$ and $\lambda \in \Omega_\epsilon := \{\lambda \in \mathbb{C} \mid \Re \lambda \geq -\hat{\beta}\epsilon^3\}$. Then there exist positive constants C and ϵ_0 such that if $\epsilon \in (0, \epsilon_0)$ and $\lambda \in \Omega_\epsilon$,*

$$(4.35) \quad \|(\lambda - \lambda_+(D))^{-1}\|_{B(L_\alpha^2)} \leq C\epsilon^{-1},$$

$$(4.36) \quad \|(\lambda - \lambda_-(D))^{-1}\|_{B(L_\alpha^2)} \leq C\epsilon^{-3},$$

$$(4.37) \quad \|(\lambda - \lambda_-(D))^{-1}\|_{B(Y_{high})} \leq CK^{-2}\epsilon^{-3},$$

$$(4.38) \quad \|B^{-1}\mu(D)(\lambda - \lambda_-(D))^{-1}\|_{B(Y_{high})} + \|B^{-1}\partial_z(\lambda - \lambda_-(D))^{-1}\|_{B(Y_{high})} \leq CK^{-1}\epsilon^{-2},$$

$$(4.39) \quad \|B^{-1}\mu(D)(\lambda - \lambda_-(D))^{-1}\|_{B(Y_{low})} + \|B^{-1}\partial_z(\lambda - \lambda_-(D))^{-1}\|_{B(Y_{low})} \leq C\epsilon^{-2}.$$

Proof. By (4.9) and (4.11),

$$(4.40) \quad \inf_{\lambda \in \Omega_\epsilon, (\xi, \eta) \in \mathbb{R}^2} |\lambda - \lambda_+(\xi + i\alpha, \eta)| \geq \inf_{\lambda \in \Omega_\epsilon, (\xi, \eta) \in \mathbb{R}^2} \Re(\lambda - \lambda_+(\xi + i\alpha, \eta)) \gtrsim \epsilon,$$

$$\inf_{\lambda \in \Omega_\epsilon, (\xi, \eta) \in \mathbb{R}^2} |\lambda - \lambda_-(\xi + i\alpha, \eta)| \geq \inf_{\lambda \in \Omega_\epsilon, (\xi, \eta) \in \mathbb{R}^2} \left(\Re \lambda + \frac{\alpha}{2}(c-1) \right) \gtrsim \epsilon^3.$$

Hence it follows from (3.5) that

$$\begin{aligned} \|(\lambda - \lambda_+(D))^{-1}\|_{B(L_\alpha^2)} &= \sup_{(\xi, \eta) \in \mathbb{R}^2} \frac{1}{|\lambda - \lambda_+(\xi + i\alpha, \eta)|} \leq C\epsilon^{-1}, \\ \|(\lambda - \lambda_-(D))^{-1}\|_{B(L_\alpha^2)} &= \sup_{(\xi, \eta) \in \mathbb{R}^2} \frac{1}{|\lambda - \lambda_-(\xi + i\alpha, \eta)|} \leq C\epsilon^{-3}, \end{aligned}$$

where C is a positive constants that does not depend on $\epsilon \in (0, \epsilon_0)$ and $\lambda \in \Omega_\epsilon$.

Next, we will show (4.37). Suppose $f \in Y_{high}$. Then $\text{supp } \hat{f}(\xi + i\alpha, \eta) \subset \tilde{A}_{low}^c \subset A_{\xi, m} \cup A_{\eta, m} \cup A_{high}$. By Lemma 4.5,

$$(4.41) \quad \inf_{\lambda \in \Omega_\epsilon, (\xi, \eta) \in A_{\xi, m} \cup A_{\eta, m} \cup A_{high}} |\lambda - \lambda_-(\xi + i\alpha, \eta)| \gtrsim K^2\epsilon^3.$$

Hence it follows from (3.5) that

$$\begin{aligned} \|(\lambda - \lambda_-(D))^{-1}f\|_{L_\alpha^2(\mathbb{R}^2)} &\leq \sup_{(\xi, \eta) \notin \tilde{A}_{low}} \frac{1}{|\lambda - \lambda_-(\xi + i\alpha, \eta)|} \left(\int_{\mathbb{R}^2} |\hat{f}(\xi + i\alpha, \eta)|^2 d\xi d\eta \right)^{1/2} \\ &\lesssim K^{-2}\epsilon^{-3} \|f\|_{L_\alpha^2(\mathbb{R}^2)}. \end{aligned}$$

Next, we will prove (4.38). By (4.28),

$$(4.42) \quad \sup_{\lambda \in \Omega_\epsilon, (\xi, \eta) \in A_{high}} \frac{|\xi + i\alpha| + |\mu(\xi + i\alpha, \eta)|}{|B(\xi + i\alpha, \eta)|^{1/2} |\lambda - \lambda_-(\xi + i\alpha, \eta)|} \lesssim \delta^{-2}\epsilon^{-1},$$

where $B(\xi, \eta) = 1 + b(\xi^2 + \eta^2)$. By (4.26) and the definition of $A_{\xi, m}$,

$$(4.43) \quad \frac{|\xi + i\alpha| + |\mu(\xi + i\alpha, \eta)|}{|\lambda - \lambda_-(\xi + i\alpha, \eta)|} \lesssim \frac{\sqrt{\xi^2 + \eta^2}}{\epsilon \xi^2} \lesssim \frac{1}{K\epsilon^2} \quad \text{for } (\xi, \eta) \in A_{\xi, m} \text{ and } \lambda \in \Omega_\epsilon.$$

By (4.27) and the fact that $|\xi + i\alpha| + |\mu(\xi + i\alpha, \eta)| \lesssim K\epsilon$ for $(\xi, \eta) \in A_{\eta, m}$,

$$(4.44) \quad \frac{|\xi + i\alpha| + |\mu(\xi + i\alpha, \eta)|}{|\lambda - \lambda_-(\xi + i\alpha, \eta)|} \lesssim K|\xi + i\alpha|^2\eta^{-2} \lesssim \frac{1}{K\epsilon^2} \quad \text{for } (\xi, \eta) \in A_{\eta, m} \text{ and } \lambda \in \Omega_\epsilon.$$

Combining (4.42)–(4.44) with

$$(4.45) \quad |B(\xi + i\alpha, \eta)| \geq 1 - b\alpha^2 > 0,$$

we have (4.38).

Finally, we will prove (4.39). By (4.25), we have for $\lambda \in \Omega_\epsilon$ and $(\xi, \eta) \in \tilde{A}_{low}$,

$$\begin{aligned} & |\lambda - \lambda_-(\xi + i\alpha, \eta)| \\ & \geq \epsilon^3 \left[-\hat{\beta} + \frac{\hat{\alpha}}{2} \{1 - (b-a)\hat{\alpha}^2 + 3(b-a)\hat{\xi}^2 + \frac{\hat{\eta}^2}{\hat{\xi}^2 + \hat{\alpha}^2}\} \right] + O(K^9 \epsilon^5) \\ & \gtrsim \epsilon^3 (1 + \hat{\xi}^2). \end{aligned}$$

$$\sup_{\lambda \in \Omega_\epsilon, (\xi, \eta) \in \tilde{A}_{low}} \frac{|\xi + i\alpha|}{|\lambda - \lambda_-(\xi + i\alpha, \eta)|} \lesssim \sup_{\hat{\xi} \in [0, K]} \frac{|\hat{\xi}| + \hat{\alpha}}{\epsilon^2 (1 + \hat{\xi}^2)} = O(\epsilon^{-2}).$$

Thus we complete the proof. \square

5. SPECTRAL STABILITY FOR SMALL LINE SOLITARY WAVES

In this section, we will prove Theorem 2.4. For small line solitary waves, the spectrum of the linearized operator \mathcal{L} is well approximated by that of \mathcal{L}_{KP} in the low frequency regime, while the spectrum of \mathcal{L} is close to that of the free operator \mathcal{L}_0 in the high-frequency regime. We will show that any spectrum of \mathcal{L} locates in the stable half plane and is bounded away from the imaginary axis except for the continuous eigenvalues $\{\lambda_\epsilon(\eta)\}$. More precisely, we will prove

$$(5.1) \quad \sup_{\lambda \in \Omega_\epsilon} \|(\lambda - \mathcal{L})^{-1} \mathcal{Q}(\epsilon^2 \eta_0)\|_{B(X)} < \infty.$$

Since the potential part of \mathcal{L} is independent of y , we can estimate the high frequency part in y and the low frequency part in y , separately.

5.1. Spectral stability for high frequencies in y . First, we will estimate solutions of the resolvent equation

$$(5.2) \quad (\lambda - \mathcal{L})u = f$$

for $f \in \tilde{\rho}_y(D_y)X$. In the high frequency regime in y , the potential term V is relatively small compared with $\lambda - \mathcal{L}_0$.

Lemma 5.1. *Let c , α , $\hat{\alpha}$ and Ω_ϵ be as in Lemma 4.6. There exists a positive number ϵ_0 such that if $\epsilon \in (0, \epsilon_0)$ and $\lambda \in \Omega_\epsilon$, then*

$$\sup_{\lambda \in \Omega_\epsilon} \|\tilde{\rho}_y(D_y)(\lambda - \mathcal{L})^{-1} \tilde{\rho}_y(D_y)\|_{B(X)} < \infty.$$

Proof of Lemma 5.1. In view of Lemma 4.1 and the second resolvent formula

$$(5.3) \quad (\lambda - \mathcal{L})^{-1} = \{I - (\lambda - \mathcal{L}_0)^{-1}V\}^{-1}(\lambda - \mathcal{L}_0)^{-1},$$

it suffices to show that

$$(5.4) \quad \sup_{\lambda \in \Omega_\epsilon} \|\tilde{\rho}_y(D_y)(I - (\lambda - \mathcal{L}_0)^{-1}V)^{-1}\tilde{\rho}_y(D_y)\|_{B(X)} < \infty.$$

By (4.20),

$$\begin{aligned} (\lambda - \mathcal{L}_0)^{-1}V &= -(\lambda - \lambda_+(D))^{-1}(\lambda - \lambda_-(D))^{-1}B^{-1} \begin{pmatrix} v_1 & v_2 \\ (\lambda - c\partial_z)v_1 & (\lambda - c\partial_z)v_2 \end{pmatrix} \\ &=: \begin{pmatrix} r_{11}(\lambda) & r_{12}(\lambda) \\ r_{21}(\lambda) & r_{22}(\lambda) \end{pmatrix}. \end{aligned}$$

First, we will show (5.4) admitting

$$\begin{aligned} (5.5) \quad & \sup_{\lambda \in \Omega_\epsilon} (\|\tilde{\rho}_y(D_y)r_{11}(\lambda)\tilde{\rho}_y(D_y)\|_{B(H_\alpha^1)} + \|\tilde{\rho}_y(D_y)r_{22}(\lambda)\tilde{\rho}_y(D_y)\|_{B(L_\alpha^2)}) = O(K^{-1}), \\ & \sup_{\lambda \in \Omega_\epsilon} \|\tilde{\rho}_y(D_y)r_{12}(\lambda)\tilde{\rho}_y(D_y)\|_{B(L_\alpha^2, H_\alpha^1)} = O(K^{-1}\epsilon^{-1} + \delta^{-2}), \\ & \sup_{\lambda \in \Omega_\epsilon} \|\tilde{\rho}_y(D_y)r_{21}(\lambda)\tilde{\rho}_y(D_y)\|_{B(H_\alpha^1, L_\alpha^2)} = O(\epsilon\delta^{-2}). \end{aligned}$$

Let

$$\begin{aligned} B_1(\lambda) &= \begin{pmatrix} I - r_{11}(\lambda) & -r_{12}(\lambda) \\ O & I - r_{22}(\lambda) \end{pmatrix}, \\ B_2(\lambda) &= \begin{pmatrix} I - (I - r_{11}(\lambda))^{-1}r_{12}(\lambda)(I - r_{22}(\lambda))^{-1}r_{21}(\lambda) & 0 \\ -(I - r_{22}(\lambda))^{-1}r_{21}(\lambda) & I \end{pmatrix}. \end{aligned}$$

Then $I - (\lambda - \mathcal{L}_0)^{-1}V = B_1(\lambda)B_2(\lambda)$. We see from (5.5) that $I - r_{ii}(\lambda)$ ($i = 1, 2$) have bounded inverse and that

$$\begin{aligned} & \|\tilde{\rho}_y(D_y)r_{12}(\lambda)\tilde{\rho}_y(D_y)\|_{B(L_\alpha^2, H_\alpha^1)} \|\tilde{\rho}_y(D_y)r_{21}(\lambda)\tilde{\rho}_y(D_y)\|_{B(H_\alpha^1, L_\alpha^2)} \\ &= O(K^{-1}\delta^{-2} + \epsilon\delta^{-4}) = O(\epsilon^{1/20}). \end{aligned}$$

Thus we have

$$\sup_{\lambda \in \Omega_\epsilon} (\|\tilde{\rho}_y(D_y)B_1(\lambda)^{-1}\tilde{\rho}_y(D_y)\|_{B(X)} + \|\tilde{\rho}_y(D_y)B_1(\lambda)^{-1}\tilde{\rho}_y(D_y)\|_{B(X)}) < \infty,$$

and $\tilde{\rho}_y(D_y)(I - (\lambda - \mathcal{L}_0)^{-1}V)^{-1}\tilde{\rho}_y(D_y) \in L^\infty(\Omega_\epsilon; B(X))$.

Now we will start to show (5.5). By (4.21),

$$(5.6) \quad \Delta(\lambda - \lambda_+(D))^{-1}(\lambda - \lambda_-(D))^{-1} = \frac{i\mu(D)}{2S(D)} \{(\lambda - \lambda_+(D))^{-1} - (\lambda - \lambda_-(D))^{-1}\},$$

$$(5.7) \quad (\lambda - c\partial_z)(\lambda - \lambda_+(D))^{-1}(\lambda - \lambda_-(D))^{-1} = \frac{1}{2} \{(\lambda - \lambda_+(D))^{-1} + (\lambda - \lambda_-(D))^{-1}\}.$$

If $|\eta| \geq K(K + \hat{\alpha})\epsilon^2$, then $(\xi, \eta) \in \tilde{A}_{low}^c \subset A_{high} \cup A_{\xi, m} \cup A_{\eta, m}$. Since

$$(5.8) \quad v_{1,c} = cq_c'' - c\Delta(q_c \cdot),$$

it follows from (5.6), (A.5) and Claim A.1 in Appendix A that

$$\|\tilde{\rho}_y(D_y)(\lambda - \lambda_+(D))^{-1}(\lambda - \lambda_-(D))^{-1}B^{-1}v_{1,c}\|_{B(H_\alpha^1)} \lesssim I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \epsilon^4 \|\tilde{\rho}_y(D_y)(\lambda - \lambda_+(D))^{-1}(\lambda - \lambda_-(D))^{-1}B^{-1}\|_{B(L_\alpha^2)}, \\ I_2 &= \epsilon^2 \sum_{\pm} \|\tilde{\rho}_y(D_y)(\lambda - \lambda_{\pm}(D))^{-1}B^{-1}\mu(D)\|_{B(L_\alpha^2)}. \end{aligned}$$

By (4.40), (4.41) and (4.45),

$$I_1 = \epsilon^4 \sup_{(\xi, \eta) \notin \tilde{A}_{low}} \frac{|B(\xi + i\alpha, \eta)|^{-1}}{|\lambda - \lambda_+(\xi + i\alpha, \eta)| |\lambda - \lambda_-(\xi + i\alpha, \eta)|} = O(K^{-2}).$$

By (4.40), (4.42)–(4.44) and Claim A.3,

$$I_2 \lesssim \epsilon + \epsilon^2 \sup_{(\xi, \eta) \notin \tilde{A}_{low}} \frac{|\mu(\xi + i\alpha, \eta)|}{|B(\xi + i\alpha, \eta)|} |\lambda - \lambda_-(\xi + i\alpha, \eta)|^{-1} \lesssim K^{-1}.$$

Thus we prove

$$\sup_{\lambda \in \Omega_\epsilon} \|\tilde{\rho}_y(D_y)r_{11}(\lambda)\tilde{\rho}_y(D_y)\|_{B(H_\alpha^1)} \lesssim I_1 + I_2 = O(K^{-1}).$$

Next, we will estimate $\tilde{\rho}_y(D_y)r_{12}(\lambda)\tilde{\rho}_y(D_y)$. Since

$$(5.9) \quad v_{2,c} = 2\partial_z(q_c \cdot) - q'_c,$$

we have from Claim A.1

$$\|\tilde{\rho}_y(D_y)r_{12}(\lambda)\tilde{\rho}_y(D_y)\|_{B(L_\alpha^2, H_\alpha^1)} \lesssim I_3 + I_4,$$

where

$$\begin{aligned} I_3 &= \epsilon^2 \|\tilde{\rho}_y(D_y)(\lambda - \lambda_+(D))^{-1}(\lambda - \lambda_-(D))^{-1}\partial_z B^{-1}\|_{B(L_\alpha^2, H_\alpha^1)}, \\ I_4 &= \epsilon^3 \|\tilde{\rho}_y(D_y)(\lambda - \lambda_+(D))^{-1}(\lambda - \lambda_-(D))^{-1}B^{-1}\|_{B(L_\alpha^2, H_\alpha^1)}. \end{aligned}$$

By (3.5),

$$\begin{aligned} I_3 &\lesssim \sup_{(\xi, \eta) \notin \tilde{A}_{low}} \frac{\epsilon^2 |\xi + i\alpha|}{|\lambda - \lambda_+(\xi + i\alpha, \eta)| |\lambda - \lambda_-(\xi + i\alpha, \eta)| |B(\xi + i\alpha, \eta)|^{1/2}}, \\ I_4 &\lesssim \sup_{(\xi, \eta) \notin \tilde{A}_{low}} \frac{\epsilon^3}{|\lambda - \lambda_+(\xi + i\alpha, \eta)| |\lambda - \lambda_-(\xi + i\alpha, \eta)|}. \end{aligned}$$

It follows from (4.40) and (4.42)–(4.45) that

$$\begin{aligned} I_3 &\lesssim \sup_{(\xi, \eta) \in A_{high}} \frac{\epsilon^2 |\xi + i\alpha|}{|\lambda - \lambda_+(\xi + i\alpha, \eta)| |\lambda - \lambda_-(\xi + i\alpha, \eta)| |B(\xi + i\alpha, \eta)|^{1/2}} \\ &\quad + \sup_{(\xi, \eta) \in A_{\xi, m} \cup A_{\eta, m}} \frac{\epsilon^2 |\xi + i\alpha|}{|\lambda - \lambda_+(\xi + i\alpha, \eta)| |\lambda - \lambda_-(\xi + i\alpha, \eta)|} \\ &\lesssim \delta^{-2} + K^{-1} \epsilon^{-1}. \end{aligned}$$

By (4.40) and (4.41),

$$I_4 = O(K^{-2}\epsilon^{-1}).$$

Thus we prove

$$\sup_{\lambda \in \Omega_\epsilon} \|\tilde{\rho}_y(D_y)r_{12}(\lambda)\tilde{\rho}_y(D_y)\|_{B(L_\alpha^2)} \lesssim K^{-1}\epsilon^{-1} + \delta^{-2}.$$

Using (5.7), we can estimate r_{21} and r_{22} in exactly the same way. Thus we complete the proof. \square

5.2. Spectral stability for low frequencies in y . Now we will estimate solutions of (5.2) for $f \in \rho_y(D_y)X$ satisfying the orthogonality condition

$$(5.10) \quad \int_{\mathbb{R}} (\mathcal{F}_y f)(x, \eta) \cdot \overline{g_k^*(x, \eta)} dx = 0 \quad \text{for } \eta \in [-\epsilon^2\eta_0, \epsilon^2\eta_0] \text{ and } k = 1, 2.$$

Let $\tilde{f} = (\tilde{f}_1, \tilde{f}_2)$ and $f = (f_1, f_2) = P(D)\tilde{f}$. To begin with, We will show that (5.10) is reduced to the secular term condition that \tilde{f}_2 does not include the resonant modes of the linearized KP-II operator \mathcal{L}_{KP} in the limit $\epsilon \rightarrow 0$.

Let $E_\epsilon : L_\alpha^2(\mathbb{R}^2) \rightarrow L_\alpha^2(\mathbb{R}^2)$ be an isomorphism defined by $(E_\epsilon f)(x, y) := \epsilon^{-3/2}f(\epsilon^{-1}x, \epsilon^{-2}y)$ and let

$$\begin{aligned} Z &= \{f \in \rho_y(D_y)X \mid \mathcal{P}(\epsilon^2\eta_0)f = 0\}, \quad \tilde{Z} = P(D)^{-1}Z, \\ \overline{Z} &= \{(\bar{f}_1, \bar{f}_2) \in Y \times Y \mid \mathcal{P}_{KP}(\epsilon^2\eta_0)E_\epsilon \rho_z(D_z)\bar{f}_2 = 0\}. \end{aligned}$$

Note that $P(D) : Y \times Y \rightarrow \rho_y(D_y)X$ is isomorphic for small $\epsilon > 0$ because $|\mu(\xi + i\alpha, \eta)|$ is bounded away from 0 for $\eta \in \text{supp } \rho_y$. Let $\overline{P}(\eta_0)$ be the projection on $L^2(\mathbb{R}^2; \mathbb{C}^2)$ defined by

$$\overline{P}(\eta_0) \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \rho_z(D_z)E_\epsilon^{-1}\mathcal{P}_{KP}(\eta_0)E_\epsilon \rho_z(D_z)\tilde{u}_2 \end{pmatrix}.$$

The subspaces \tilde{Z} and \overline{Z} are isomorphic provided ϵ is small.

Lemma 5.2. *Let ϵ_0 and η_0 be sufficiently small positive numbers. Then for $\epsilon \in (0, \epsilon_0)$, there exists an operator $\Pi : \tilde{Z} \rightarrow \overline{Z}$ such that*

$$\|\Pi - I\|_{B(\tilde{Z}, \overline{Z})} + \|\Pi^{-1} - I\|_{B(\overline{Z}, \tilde{Z})} = O(K^{-1}).$$

Let $W_1 = H_\alpha^1(\mathbb{R}) \times L_\alpha^2(\mathbb{R})$, $W_0 = L_\alpha^2(\mathbb{R}; \mathbb{C}^2)$, $W_0^* = L_{-\alpha}^2(\mathbb{R}; \mathbb{C}^2)$ and $W_1^* = H_{-\alpha}^{-1}(\mathbb{R}) \times L_{-\alpha}^2(\mathbb{R})$. To prove Lemma 5.2, we need the following.

Claim 5.3. *Let $\hat{\alpha} \in (0, \hat{\alpha}_0)$, $\alpha = \hat{\alpha}\epsilon$ and let ϵ_0 and η_0 be sufficiently small positive numbers. If $\epsilon \in (0, \epsilon_0)$ and $\eta \in [-\eta_0, \eta_0]$, then*

$$\begin{aligned} \left\| P(D_z, \epsilon^2\eta)^{-1} - \frac{1}{2} \begin{pmatrix} \partial_z & S^{-1}(D_z, \epsilon^2\eta) \\ -\partial_z & S^{-1}(D_z, \epsilon^2\eta) \end{pmatrix} \right\|_{B(W_1, W_0)} &= O(\epsilon^2\eta^2), \\ \left\| P^*(D_z, \epsilon^2\eta) - \begin{pmatrix} (\partial_z^*)^{-1} & \overline{S}^{-1}(D_z, \epsilon^2\eta) \\ -(\partial_z^*)^{-1} & \overline{S}^{-1}(D_z, \epsilon^2\eta) \end{pmatrix} \right\|_{B(W_1^*, W_0^*)} &= O(\epsilon^2\eta^2), \end{aligned}$$

where $(\partial_z^*)^{-1}f(z) = -\int_{-\infty}^z f(z_1) dz_1$.

Proof. In view of (A.1), (A.4) and their proofs,

$$(5.11) \quad \|i\mu(D_z, \epsilon^2\eta) - \partial_z\|_{B(W_1)} + \|i\bar{\mu}(D_z, \epsilon^2\eta)^{-1} - (\partial_z^*)^{-1}\|_{B(W_1^*)} = O(\epsilon^2\eta^2).$$

Since

$$P(\xi, \eta)^{-1} = \frac{1}{2} \begin{pmatrix} i\mu(\xi, \eta) & S(\xi, \eta)^{-1} \\ -i\mu(\xi, \eta) & S(\xi, \eta)^{-1} \end{pmatrix}, \quad P^*(\xi, \eta) = \begin{pmatrix} \overline{i\mu(\xi, \eta)}^{-1} & \overline{S(\xi, \eta)} \\ -\overline{i\mu(\xi, \eta)}^{-1} & \overline{S(\xi, \eta)} \end{pmatrix},$$

Claim 5.3 follows from (5.11). \square

Proof of Lemma 5.2. Let $\Pi\tilde{u} = \tilde{u} - \bar{P}(\eta_0)\tilde{u}$ for $\tilde{u} \in \tilde{Z}$. To prove Lemma 5.2, it suffices to show

$$(5.12) \quad \|P(D)\mathcal{P}(\epsilon^2\eta_0)P(D)^{-1} - \bar{P}(\eta_0)\|_Y = O(K^{-1}).$$

See e.g. [20, Chapter I, Section 4.6].

First, we will show

$$(5.13) \quad \|\mathcal{P}(\epsilon^2\eta_0)\tilde{\rho}_z(D_z)f\|_{L_\alpha^2(\mathbb{R}^2)} + \|\tilde{\rho}_z(D_z)\mathcal{P}(\epsilon^2\eta_0)f\|_{L_\alpha^2(\mathbb{R}^2)} \lesssim K^{-1}\|f\|_{L_\alpha^2(\mathbb{R}^2)}.$$

Let

$$\tilde{c}_k(\eta) = \int \tilde{\rho}_z(D_z)\mathcal{F}_y f(z, \eta) \cdot g_k^*(z, \eta) dz.$$

Then

$$\mathcal{P}(\epsilon^2\eta_0)\tilde{\rho}_z(D_z)f = \frac{1}{\sqrt{2\pi}} \sum_{k=1,2} \int_{-\epsilon^2\eta_0}^{\epsilon^2\eta_0} \tilde{c}_k(\eta)g_k(z, \eta)e^{iy\eta} d\eta.$$

Since $\|\partial_z^{-1}\tilde{\rho}_z(D_z)\|_{B(L_\alpha^2(\mathbb{R}))} \leq (K\epsilon)^{-1}$ and $\sup_{\eta \in [-\epsilon^2\eta_0, \epsilon^2\eta_0]} \|\partial_z g_k^*(\cdot, \eta)\|_{L_{-\alpha}^2(\mathbb{R})} = O(\epsilon)$ by Corollary 3.5,

$$\begin{aligned} |\tilde{c}_k(\eta)| &= \left| \int \partial_z^{-1}\tilde{\rho}_z(D_z)(\mathcal{F}_y f)(z, \eta) \cdot \partial_z g_k^*(z, \eta) dz \right| \\ &\lesssim K^{-1}\|\mathcal{F}_y f(\cdot, \eta)\|_{L_\alpha^2(\mathbb{R}_z)}. \end{aligned}$$

Hence it follows from the Plancherel theorem and the above that

$$\begin{aligned} \|\mathcal{P}(\epsilon^2\eta_0)\tilde{\rho}_z(D_z)f\|_{L_\alpha^2(\mathbb{R}^2)} &\lesssim \sum_{k=1,2} \left\| \|c_k(\eta)g_k(x, \eta)\|_{L_\alpha^2(\mathbb{R}_z)} \right\|_{L^2(-\epsilon^2\eta_0 \leq \eta \leq \epsilon^2\eta_0)} \\ &\lesssim K^{-1}\|f\|_{L_\alpha^2(\mathbb{R}^2)}. \end{aligned}$$

Similarly, we have $\|\tilde{\rho}_z(D_z)\mathcal{P}(\epsilon^2\eta_0)f\|_{L_\alpha^2(\mathbb{R}^2)} \lesssim K^{-1}\|f\|_{L_\alpha^2(\mathbb{R}^2)}$. Thus we prove (5.13).

Next, we will show $\rho_z(D_z)P(\epsilon^2\eta_0)\rho_z(D_z) \simeq \rho_z(D_z)E_\epsilon^{-1}P_{KP}(\eta_0)E_\epsilon\rho_z(D_z)$. By the fact that $\rho_z(D_z)$ is bounded on $L_\alpha^2(\mathbb{R}^2)$ and $\|f(\cdot)\|_{L_\alpha^2(\mathbb{R})} = \epsilon^{-1/2}\|f(\epsilon^{-1}\cdot)\|_{L_\alpha^2(\mathbb{R})}$,

$$(5.14) \quad \begin{aligned} &\left\| \rho_z(\epsilon D_z) \left\{ \epsilon^{-1}P(\epsilon D_z, \epsilon^2\eta)^{-1}g_k(\epsilon^{-1}\cdot, \epsilon^2\eta) - \begin{pmatrix} 0 \\ g_{0,k}(\cdot, \eta) \end{pmatrix} \right\} \right\|_{L_\alpha^2(\mathbb{R})} \\ &\leq II_1 + II_2 + II_3 + II_4 = O(K^{-2}\epsilon + \eta^2), \end{aligned}$$

where

$$\begin{aligned}
II_1 &= \epsilon^{-3/2} \left\| \left\{ P(\epsilon D_z, \epsilon^2 \eta)^{-1} - \frac{1}{2} \begin{pmatrix} \partial_z & S^{-1}(D_z, \epsilon^2 \eta) \\ -\partial_z & S^{-1}(D_z, \epsilon^2 \eta) \end{pmatrix} \right\} g_k(\cdot, \epsilon^2 \eta) \right\|_{L_\alpha^2}, \\
II_2 &= \frac{1}{2} \epsilon^{-3/2} \left\| (S^{-1}(D_z, \epsilon^2 \eta) - I) \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} g_k(\cdot, \epsilon^2 \eta) \right\|_{L_\alpha^2}, \\
II_3 &= \epsilon^{-1} \left\| \frac{1}{2} \begin{pmatrix} \partial_z & I \\ -\partial_z & I \end{pmatrix} g_k(\epsilon^{-1} \cdot, \epsilon^2 \eta) - \begin{pmatrix} 0 \\ g_{0,k}(\cdot, 0) \end{pmatrix} \right\|_{L_\alpha^2(\mathbb{R})}, \\
II_4 &= \|g_{0,k}(\cdot, \eta) - g_{0,k}(\cdot, 0)\|_{L_\alpha^2(\mathbb{R})}.
\end{aligned}$$

Indeed, it follows from Corollary 3.5 that $II_3 = O(\epsilon^2 + \eta^2)$ and that for $k = 1, 2$,

$$\|g_k(\cdot, \epsilon^2 \eta)\|_{L_\alpha^2(\mathbb{R})} = O(\epsilon^{-1/2}), \quad \left\| \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} g_k(\cdot, \eta) \right\|_{L_\alpha^2(\mathbb{R})} = O(\epsilon^{1/2}).$$

Combining the above with Claim 5.3 and (A.8), we have $II_1 = O(\eta^2)$ and $II_2 = O(K^{-2}\epsilon)$ and we have $II_4 = O(\eta^2)$ from (3.6). We can prove

$$(5.15) \quad \left\| \rho_z(\epsilon D_z) \left\{ P^*(\epsilon D_z, \epsilon^2 \eta) g_k^*(\epsilon^{-1} \cdot, \epsilon^2 \eta) - \begin{pmatrix} 0 \\ g_{0,k}^*(\cdot, \eta) \end{pmatrix} \right\} \right\|_{L_{-\hat{\alpha}}^2(\mathbb{R})} = O(\eta^2 + K^{-2}\epsilon)$$

in the same way.

Since

$$\begin{aligned}
& P(D_z, \eta)^{-1} \mathcal{P}(\epsilon^2 \eta_0) f \\
&= \sum_{k=1,2} (2\pi)^{-1/2} \int_{-\epsilon^2 \eta_0}^{\epsilon^2 \eta_0} \langle \mathcal{F}_y \tilde{f}(\cdot, \eta), P(D_z, \eta)^* g_k^*(\cdot, \eta) \rangle P(D_z, \eta)^{-1} g_k(x, \eta) e^{iy\eta} d\eta
\end{aligned}$$

for $f = P(D) \tilde{f}$, we have from (5.14) and (5.15) that

$$\begin{aligned}
(5.16) \quad & \left\| \rho_z(D_z) \left\{ P(D)^{-1} \mathcal{P}(\epsilon^2 \eta_0) P(D) - \begin{pmatrix} 0 & 0 \\ 0 & \rho_z(D_z) E_\epsilon^{-1} \mathcal{P}_{KP}(\eta_0) E_\epsilon \rho_z(D_z) \end{pmatrix} \right\} \rho_z(D_z) \right\|_{B(Y)} \\
&= O(\eta_0^2 + K^{-1}).
\end{aligned}$$

Finally, we will prove that for a $\tau_0 > 0$,

$$(5.17) \quad \|\tilde{\rho}_z(\epsilon D_z) \mathcal{P}_{KP}(\eta_0)\|_{B(L_\alpha^2)} + \|\mathcal{P}_{KP}(\eta_0) \tilde{\rho}_z(\epsilon D_z)\|_{B(L_\alpha^2)} = O(e^{-\tau_0 K}).$$

Since $\tilde{g}_0(z, \eta) := e^{\hat{\alpha}z} g_0(z, \eta)$ and $\tilde{g}_0^*(z, \eta) = e^{-\hat{\alpha}z} g_0^*(z, \eta)$ are analytic on $\{z \in \mathbb{C} \mid |\Im z| < \hat{\alpha}_0\}$ and $\sup_{\tau \in [-\tau_0, \tau_0]} (\|\tilde{g}_0(z + i\tau, \eta)\|_{L^1(\mathbb{R}_z)} + \|\tilde{g}_0^*(z + i\tau, \eta)\|_{L^1(\mathbb{R}_z)}) < \infty$ for any $\tau_0 \in [0, \hat{\alpha}_0)$ and $\eta \in [-\eta_0, \eta_0]$, it follows from the Paley-Wiener theorem that there exists a C_{τ_0} for any $\tau_0 \in [0, \hat{\alpha})$ such that

$$(5.18) \quad \sup_{\eta \in [-\eta_0, \eta_0]} (|\mathcal{F}_z \tilde{g}_0(\xi, \eta)| + |\mathcal{F}_z \tilde{g}_0^*(\xi, \eta)|) \leq C_{\tau_0} e^{-\tau_0 |\xi|}.$$

By (5.17) and the definition of $\mathcal{P}_{KP}(\eta_0)$, we have (5.17). Combining (5.13), (5.16) and (5.17), we have (5.12). Thus we complete the proof. \square

Next, we will show that $(\lambda - \mathcal{L})^{-1}|_Z$ is uniformly bounded in $\lambda \in \Omega_\epsilon$.

Lemma 5.4. *Let c , α and ϵ_0 be as in Lemma 5.1. Then there exists a positive constant C such that*

$$\sup_{\lambda \in \Omega_\epsilon} \|(\lambda - \mathcal{L})^{-1}f\|_X \leq C\|f\|_X \quad \text{for any } f \in Z.$$

Let $f \in Z$ and

$$(5.19) \quad \bar{u} = {}^t(\bar{u}_1, \bar{u}_2) := \Pi P(D)^{-1}u, \quad \bar{f} := {}^t(\bar{f}_1, \bar{f}_2) = \Pi P(D)^{-1}f.$$

Then $\bar{f} \in \bar{Z}$ and (5.2) is translated into

$$(5.20) \quad \begin{cases} (\lambda - \lambda_+(D) - a_1 - \bar{r}_{11})\bar{u}_1 - (a_2 + \bar{r}_{12})\bar{u}_2 = \bar{f}_1, \\ (\lambda - \lambda_-(D) - a_2 - \bar{r}_{22})\bar{u}_2 - (a_1 + \bar{r}_{21})\bar{u}_1 = \bar{f}_2, \end{cases}$$

where

$$\begin{aligned} a_1 &= \frac{i}{2}B^{-1}S(D)^{-1}v_{1,c}\mu(D)^{-1} - \frac{1}{2}B^{-1}S(D)^{-1}v_{2,c}S(D), \\ a_2 &= -\frac{i}{2}B^{-1}S(D)^{-1}v_{1,c}\mu(D)^{-1} - \frac{1}{2}B^{-1}S(D)^{-1}v_{2,c}S(D), \\ \begin{pmatrix} \bar{r}_{11} & \bar{r}_{12} \\ \bar{r}_{21} & \bar{r}_{22} \end{pmatrix} &= \left[\Pi, \begin{pmatrix} \lambda_+(D) + a_1 & a_2 \\ a_1 & \lambda_-(D) + a_2 \end{pmatrix} \right] \Pi^{-1}. \end{aligned}$$

We decompose \bar{f}_2 and \bar{u}_2 into the high frequency part and the low frequency part. Let $\bar{u}_{2,h} = \tilde{\rho}_z(D_z)\bar{u}_2$, $\bar{u}_{2,\ell} = \rho_z(D_z)\bar{u}_2$, $\bar{f}_{2,h} = \tilde{\rho}_z(D_z)\bar{f}_2$ and $\bar{f}_{2,\ell} = \rho_z(D_z)\bar{f}_2$. Then

$$\left\{ \lambda I - \begin{pmatrix} \lambda_+(D) & 0 \\ 0 & \lambda_-(D) \end{pmatrix} - A \right\} \begin{pmatrix} \bar{u}_1 \\ \bar{u}_{2,h} \\ \bar{u}_{2,\ell} \end{pmatrix} = \begin{pmatrix} \bar{f}_1 \\ \bar{f}_{2,h} \\ \bar{f}_{2,\ell} \end{pmatrix},$$

where

$$A = \begin{pmatrix} a_1 + \bar{r}_{11} & a_2 + \bar{r}_{12} & a_2 + \bar{r}_{12} \\ \tilde{\rho}_z(D_z)(a_1 + \bar{r}_{21}) & \tilde{\rho}_z(D_z)(a_2 + \bar{r}_{22})\tilde{\rho}_z(D_z) & \tilde{\rho}_z(D_z)(a_2 + \bar{r}_{22})\rho_z(D_z) \\ \rho_z(D_z)(a_1 + \bar{r}_{21}) & \rho_z(D_z)(a_2 + \bar{r}_{22})\tilde{\rho}_z(D_z) & \rho_z(D_z)(a_2 + \bar{r}_{22})\rho_z(D_z) \end{pmatrix}.$$

To estimate $\bar{u}_{2,h}$ and $\bar{u}_{2,\ell}$, we need the following.

Lemma 5.5. *Let $\hat{\alpha} \in (0, \hat{\alpha}_0/2)$, $\alpha = \hat{\alpha}\epsilon$ and Ω_ϵ be as in Lemma 4.6. There exists an $\epsilon_0 > 0$ such that*

$$(5.21) \quad \sup_{\lambda \in \Omega_\epsilon} \sup_{\epsilon \in (0, \epsilon_0)} \|a_2(\lambda - \lambda_-(D))^{-1}\rho_z(D_z)\|_{B(Y)} < \infty,$$

$$(5.22) \quad \|a_2(\lambda - \lambda_-(D))^{-1}\tilde{\rho}_z(D_z)\|_{B(Y)} = O(K^{-1}).$$

Lemma 5.6. *Let $\hat{\alpha} \in (0, \hat{\alpha}_0/2)$ and $\alpha = \hat{\alpha}\epsilon$. Let $\hat{\beta}$ be a small positive number and Ω_ϵ be as in Lemma 4.6. There exist positive constants ϵ_0 and η_0 such that if $\epsilon \in (0, \epsilon_0)$,*

$$\sup_{\lambda \in \Omega_\epsilon, \epsilon \in (0, \epsilon_0)} \|\rho_z(D_z)\{I - (a_2 + \bar{r}_{22})(\lambda - \lambda_-(D))^{-1}\}^{-1}\rho_z(D_z)E_\epsilon^{-1}\mathcal{Q}_{KP}(\eta_0)E_\epsilon\rho_z(D_z)\|_{B(Y)} < \infty.$$

Proof of Lemma 5.5. By (A.5) and the definition of a_2 ,

$$\begin{aligned} \|a_2(\lambda - \lambda_-(D))^{-1}\tilde{\rho}_z(D_z)\|_{B(Y)} &\lesssim \|B^{-1}v_{1,c}\mu(D)^{-1}(\lambda - \lambda_-(D))^{-1}\tilde{\rho}_z(D_z)\|_{B(Y)} \\ &\quad + \|B^{-1}v_{2,c}(\lambda - \lambda_-(D))^{-1}\tilde{\rho}_z(D_z)\|_{B(Y)}. \end{aligned}$$

Since

$$(5.23) \quad B^{-1}v_{1,c}\mu(D)^{-1} = c\{(q_c - B^{-1}[B, q_c])B^{-1}\mu(D)\} - 2cB^{-1}q'_c\partial_z\mu(D)^{-1},$$

it follows from Claims A.1–A.3 and Lemma 4.6 that

$$\begin{aligned} &\|B^{-1}v_{1,c}\mu(D)^{-1}(\lambda - \lambda_-(D))^{-1}\tilde{\rho}_z(D_z)\|_{B(Y)} \\ &\lesssim \epsilon^2\|B^{-1}\mu(D)(\lambda - \lambda_-(D))^{-1}\|_{B(Y_{high})} + \epsilon^3\|(\lambda - \lambda_-(D))^{-1}\|_{B(Y_{high})} \\ &= O(K^{-1}). \end{aligned}$$

We can prove

$$\begin{aligned} &\|B^{-1}v_{2,c}(\lambda - \lambda_-(D))^{-1}\tilde{\rho}_z(D_z)\|_{B(Y)} \\ &\lesssim \epsilon^3\|(\lambda - \lambda_-(D))^{-1}\|_{B(Y_{high})} + \epsilon^2\|B^{-1}\partial_z(\lambda - \lambda_-(D))^{-1}\|_{B(Y_{high})} \\ &= O(K^{-1}) \end{aligned}$$

in the same way. Thus we prove (5.22).

Next, we will show (5.21). As in the proof of (5.22), we have

$$\begin{aligned} \|a_2(\lambda - \lambda_-(D))^{-1}\rho_z(D_z)\|_{B(Y)} &\lesssim \epsilon^2\|\mu(D)(\lambda - \lambda_-(D))^{-1}\|_{B(Y_{low})} \\ &\quad + \epsilon^2\|\partial_z(\lambda - \lambda_-(D))^{-1}\|_{B(Y_{low})} + \epsilon^3\|(\lambda - \lambda_-(D))^{-1}\|_{B(Y)}. \end{aligned}$$

Combining the above with Lemma 4.6, we have (5.21). Thus we complete the proof. \square

Proof of Lemma 5.6. To prove Lemma 5.6, we approximate $\lambda_-(D) + a_2$ by \mathcal{L}_{KP} and apply Proposition 3.2. Let $E_\epsilon : L_\alpha^2(\mathbb{R}^2) \rightarrow L_\alpha^2(\mathbb{R}^2)$ be an isomorphism defined by $(E_\epsilon f)(x, y) := \epsilon^{-3/2}f(x/\epsilon, y/\epsilon^2)$, $a_{2,\epsilon} = \epsilon^{-3}E_\epsilon a_2 E_\epsilon^{-1}$ and $\lambda_{-, \epsilon}(\xi, \eta) = \epsilon^{-3}\lambda_-(\epsilon\xi, \epsilon^2\eta)$. Then

$$\left\| \rho_z(D_z) \left\{ a_2(\lambda - \lambda_-(D))^{-1} + \frac{3}{2}E_\epsilon^{-1}\partial_z(\theta_0 \cdot)(\Lambda - \mathcal{L}_{KP,0})^{-1}E_\epsilon \right\} \rho_z(D_z) \right\|_{B(Y)} \leq III_1 + III_2,$$

where $\rho_{KP}(\xi, \eta) = \rho_z(\epsilon\xi)\rho_y(\epsilon^2\eta)$ and

$$\begin{aligned} III_1 &= \left\| \rho_{KP}(D) \left\{ a_{2,\epsilon} + \frac{3}{2}\partial_z(\theta_0 \cdot) \right\} \rho_{KP}(D) \right\|_{B(L_\alpha^2)} \|(\Lambda - \lambda_{-, \epsilon}(D))^{-1}\|_{B(L_\alpha^2)} \\ III_2 &= \frac{3}{2} \left\| \rho_{KP}(D) \partial_z(\theta_0 \cdot) \{ (\Lambda - \mathcal{L}_{KP,0})^{-1} - (\Lambda - \lambda_{-, \epsilon}(D))^{-1} \} \rho_{KP}(D) \right\|_{B(L_\alpha^2)}. \end{aligned}$$

By (4.36) and (A.11), we have $III_1 = O(K^5\epsilon^2)$. By (3.5),

$$III_2 \lesssim \sup_{(\xi, \eta) \in \tilde{A}_{low}} \frac{(1 + |\xi + i\hat{\alpha}|)|\lambda_{-, \epsilon}(\xi + i\hat{\alpha}, \eta) - \mathcal{L}_{KP,0}(\xi + i\hat{\alpha}, \eta)|}{|\Lambda - \mathcal{L}_{KP,0}(\xi + i\hat{\alpha}, \eta)||\Lambda - \lambda_{-, \epsilon}(\xi + i\hat{\alpha}, \eta)|}.$$

Since

$$|\lambda_{-, \epsilon}(\xi + i\hat{\alpha}, \eta) - \mathcal{L}_{KP,0}(\xi + i\hat{\alpha}, \eta)| = O(K^8\epsilon^2)$$

by (4.25) and $\sup_{\Re \Lambda \geq -\hat{\beta}/2, (\xi, \eta) \in \mathbb{R}^2} (1 + |\xi|) |\Lambda - \mathcal{L}_{KP,0}(\xi + i\hat{\alpha}, \eta)|^{-1} < \infty$ thanks to Lemma 3.1, we have

$$III_2 \lesssim K^8 \epsilon^2.$$

Thus we have

$$(5.24) \quad \left\| \rho_z(D_z) a_2 (\lambda - \lambda_-(D))^{-1} \rho_z(D_z) + \frac{3}{2} E_\epsilon^{-1} \partial_z \{ \theta_0 (\Lambda - \mathcal{L}_{KP,0})^{-1} \} E_\epsilon \right\|_{B(Y)} = O(K^8 \epsilon^2).$$

By Lemma 4.6 and Claim A.5, we have

$$\|\bar{r}_{22}(\lambda - \lambda_-(D))^{-1}\|_Y = O(K^5 \epsilon^2).$$

Combining the above with Proposition 3.2 and (5.24), we obtain Lemma 5.6. Thus we complete the proof. \square

Now we are in position to prove Lemma 5.4.

Proof of Lemma 5.4. By Lemma 4.6, Claims A.4 and A.5,

$$(5.25) \quad \begin{aligned} & \|(\lambda - \lambda_+(D) - a_1 - \bar{r}_{11})^{-1}\|_{B(Y)} = O(\epsilon^{-1}), \\ & \|\bar{u}_1\|_Y \lesssim \epsilon^{-1} \|\bar{f}_1\|_Y + \epsilon (\|\bar{u}_{2,h}\|_Y + \|\bar{u}_{2,\ell}\|_Y). \end{aligned}$$

Since

$$\|\rho_z(D_z)(\lambda - \lambda_-(D) - a_2 - \bar{r}_{22})^{-1} \rho_z(D_z) E_\epsilon^{-1} \mathcal{Q}_{KP}(\eta_0) E_\epsilon \rho_z(D_z)\|_{B(Y)} = O(\epsilon^{-3})$$

by Lemmas 4.6 and 5.6,

$$\|\bar{u}_{2,\ell}\|_Y \lesssim \epsilon^{-3} \|\bar{f}_{2,\ell}\|_Y + K(\|\bar{u}_{2,h}\|_Y + \|\bar{u}_1\|_Y)$$

follows from Claims A.4 and A.5. Furthermore, Lemmas 5.5, 5.6 and Claim A.5 imply

$$\begin{aligned} & \|(a_2 + \bar{r}_{22}) \rho_z(D_z) (\lambda - \lambda_-(D) - a_2 - \bar{r}_{22})^{-1} \rho_z(D_z) E_\epsilon^{-1} \mathcal{Q}_{KP}(\eta_0) E_\epsilon \rho_z(D_z)\|_{B(Y)} = O(1), \\ & \|(a_2 + \bar{r}_{22}) \bar{u}_{2,\ell}\|_Y \lesssim \|\bar{f}_{2,\ell}\|_Y + K \epsilon^3 (\|\bar{u}_1\|_Y + \|\bar{u}_{2,h}\|_Y). \end{aligned}$$

By Lemma 4.6, Claims A.4 and A.5,

$$\|\tilde{\rho}_z(D_z)(\lambda - \lambda_-(D_z) - a_2 - \bar{r}_{22})^{-1} \tilde{\rho}_z(D_z)\|_{B(Y)} = O(K^{-2} \epsilon^{-3}),$$

and

$$\begin{aligned} \|\bar{u}_{2,h}\|_Y & \lesssim K^{-2} \epsilon^{-3} (\|\bar{f}_{2,h}\|_Y + \|(a_1 + \bar{r}_{21}) \bar{u}_1\|_Y + \|(a_{22} + \bar{r}_{22}) \bar{u}_{2,\ell}\|_Y) \\ & \lesssim K^{-2} \epsilon^{-3} (\|\bar{f}_{2,h}\|_Y + \|\bar{f}_{2,\ell}\|_Y) + K^{-2} \epsilon^{-1} \|\bar{u}_1\|_Y + K^{-1} \|\bar{u}_{2,h}\|_Y. \end{aligned}$$

Combining the above, we have

$$\begin{aligned} \|\bar{u}_1\|_Y & \lesssim \epsilon^{-1} \|\bar{f}_1\|_Y + \epsilon^{-2} (K^{-1} \|\bar{f}_{2,h}\|_Y + \|\bar{f}_{2,\ell}\|_Y), \\ \|\bar{u}_{2,h}\|_Y & \lesssim K^{-2} \epsilon^{-2} \|\bar{f}_1\|_Y + K^{-2} \epsilon^{-3} (\|\bar{f}_{2,h}\|_Y + \|\bar{f}_{2,\ell}\|_Y), \\ \|\bar{u}_{2,\ell}\|_Y & \lesssim K^{-1} \epsilon^{-2} \|\bar{f}_1\|_Y + \epsilon^{-3} (K^{-1} \|\bar{f}_{2,h}\|_Y + \|\bar{f}_{2,\ell}\|_Y), \end{aligned}$$

and $\sup_{\lambda \in \Omega_\epsilon} \|\Pi P(D)^{-1} (\lambda - \mathcal{L})^{-1} P(D) \Pi^{-1}\|_{B(\bar{Z})} < \infty$. Since $\Pi P(D)^{-1} : Z \rightarrow \bar{Z}$ is isomorphic, we have Lemma 5.4. Thus we complete the proof. \square

5.3. Proof of Theorem 2.4. Now we are in position to prove Theorem 2.4. Lemmas 3.3, 3.4, 5.1 and 5.4 imply (2.13) and (2.14). Taking $\hat{\beta} > 0$ smaller if necessary, we see from Gearhart-Prüss theorem that for small $\epsilon > 0$, there exists a $K = K(\epsilon)$ satisfying (2.15). This completes the proof of Theorem 2.4.

6. PROOF OF THEOREM 2.1

In this section, we will show that the eigenvalue $\lambda = 0$ of $\mathcal{L}(0)$ splits into two stable eigenvalues of $\mathcal{L}(\eta)$ for small $\eta \neq 0$ without assuming smallness of line solitary waves. As in Subsection 3.2, we will use Lyapunov Schmidt method.

To begin with, we expand $\mathcal{L}(\eta)$ as $\mathcal{L}(\eta) = \mathcal{L}(0) + \eta^2 \mathcal{L}_1(\eta)$ with

$$\mathcal{L}_1(0) = B_0^{-1} \begin{pmatrix} 0 & 0 \\ I - A_0 - B_0^{-1}A_0 + r_c & 0 \end{pmatrix} + bB_0^{-2} \begin{pmatrix} 0 & 0 \\ v_{1,c}(0) & v_{2,c}(0) \end{pmatrix}.$$

We easily see that $\|\mathcal{L}_1(\eta)\|_{B(H_\alpha^1(\mathbb{R}) \times L_\alpha^2(\mathbb{R}))} = O(1)$ as $\eta \rightarrow 0$.

Using the ansatz

$$\lambda(\eta) = i\eta\lambda_1(\eta), \quad \zeta(\eta) = \zeta_1 + \{\lambda(\eta) + \eta^2\gamma(\eta)\}\zeta_2 + \eta^2 z(\eta),$$

we will solve the eigenvalue problem (3.9). Suppose $\mathcal{L}(\eta)\zeta(\eta) = \lambda\zeta(\eta)$ and $z(\eta) \perp \zeta_1^*, \zeta_2^*$. Then

$$(6.1) \quad \mathcal{Q}_0(\mathcal{L}(\eta) - i\eta\lambda_1)z(\lambda_1, \gamma, \eta) + \mathcal{Q}_0 G(\lambda_1, \gamma, \eta) = 0,$$

$$(6.2) \quad F_k(\lambda_1, \gamma, \eta) := \langle G(\lambda_1, \gamma, \eta) + \eta^2 \mathcal{L}_1(\eta)z(\lambda_1, \gamma, \eta), \zeta_k^* \rangle = 0 \quad \text{for } k = 1, 2,$$

$$(6.3) \quad G(\lambda_1, \gamma, \eta) = \gamma(\zeta_1 - i\lambda_1\eta\zeta_2 + \eta^2 \mathcal{L}_1(\eta)\zeta_2) + \lambda_1^2 \zeta_2 + \mathcal{L}_1(\eta)(\zeta_1 + i\lambda_1\eta\zeta_2)$$

where $\mathcal{Q}_0 : H_\alpha^1(\mathbb{R}) \times L_\alpha^2(\mathbb{R}) \rightarrow \perp \ker_g(\mathcal{L}(0))^*$ is a spectral projection associated with $\mathcal{L}(0)$.

The operator $\mathcal{L}(0) : H_\alpha^2(\mathbb{R}) \times H_\alpha^1(\mathbb{R}) \rightarrow H_\alpha^1(\mathbb{R}) \times L_\alpha^2(\mathbb{R})$ is a Fredholm operator of index zero. In fact, we see from Claim 4.4, (4.20) and (4.21) with $\lambda = 0$ that $\mathcal{L}_0(0) : H_\alpha^2(\mathbb{R}) \times H_\alpha^1(\mathbb{R}) \rightarrow H_\alpha^1(\mathbb{R}) \times L_\alpha^2(\mathbb{R})$ has a bounded inverse and $V(0)$ is a compact operator on $H_\alpha^1(\mathbb{R}) \times L_\alpha^2(\mathbb{R})$. Note that $\lambda_+(D_z, 0)^{-1} \in B(L_\alpha^2(\mathbb{R}), H_\alpha^1(\mathbb{R}))$ by (4.7) and the fact that $\partial_z^{-1} \in B(L_\alpha^2(\mathbb{R}), H_\alpha^1(\mathbb{R}))$.

Thus there exist positive constants C and k such that if $|\eta|(|\lambda_1| + \|\mathcal{L}_1(\eta)\|_{B(H_\alpha^1(\mathbb{R}) \times L_\alpha^2(\mathbb{R}))}) < k$, then a solution $z = z(\lambda_1, \gamma, \eta)$ of (6.1) satisfies

$$\|z(\lambda_1, \gamma, \eta)\|_{H_\alpha^2(\mathbb{R}) \times H_\alpha^1(\mathbb{R})} \leq C \|G(\lambda_1, \gamma, \eta)\|_{H_\alpha^1(\mathbb{R}) \times L_\alpha^2(\mathbb{R})}.$$

Now we choose constants $\lambda_{1,0}$ and γ_0 so that

$$F_1(\lambda_{1,0}, \gamma_0, 0) = \gamma_0 \langle \zeta_1, \zeta_1^* \rangle + \langle \mathcal{L}_1(0)\zeta_1, \zeta_1^* \rangle + \lambda_{1,0}^2 \langle \zeta_2, \zeta_1^* \rangle = 0,$$

$$F_2(\lambda_{1,0}, \gamma_0, 0) = \langle \mathcal{L}_1(0)\zeta_1, \zeta_2^* \rangle + \lambda_{1,0}^2 \langle \zeta_2, \zeta_2^* \rangle = 0.$$

By straightforward computations, we have

$$(6.4) \quad \langle \zeta_1, \zeta_2^* \rangle = 0, \quad \langle \zeta_1, \zeta_1^* \rangle = \langle \zeta_2, \zeta_2^* \rangle = \frac{1}{2} \frac{d}{dc} E(q_c, r_c) > 0,$$

$$\begin{aligned}
(6.5) \quad \langle \zeta_2, \zeta_1^* \rangle &= - \left(\frac{c}{2} \frac{d}{dc} \int_{\mathbb{R}} q_c^2 dz + c \frac{d}{dc} \int_{\mathbb{R}} r_c dz \right) \left(\frac{d}{dc} \int_{\mathbb{R}} q_c dz \right) \\
&= \frac{16}{3c^4} \frac{bc^4 - a}{c^2 - 1} \frac{a(c^2 - 1) + (bc^2 - a) + 2c^4(2bc^2 - b - a)}{bc^2 - a} > 0,
\end{aligned}$$

$$\begin{aligned}
(6.6) \quad \langle \mathcal{L}_1(0)\zeta_1, \zeta_2^* \rangle &= \langle -A_0 q_c - B_0^{-1} A_0 q_c + q_c - bB_0^{-1} \partial_z^2 (c \frac{3}{2} q_c^2) - cq_c^2, cq_c \rangle \\
&= \left\langle \left(-\frac{4}{3} A_0 + \frac{c^2}{3} B_0 + 1 - c^2 \right) q_c, cq_c \right\rangle \\
&= -\frac{8}{15} \frac{c^2 - 1}{c} \alpha_c \{2c^2(b - a) + 3(bc^4 - a)\} < 0
\end{aligned}$$

because

$$(6.7) \quad (A_0 - c^2 B_0) q_c + \frac{3}{2} c q_c^2 = 0$$

by (1.4) and

$$\int_{\mathbb{R}} q_c(x)^2 dx = \frac{8(c^2 - 1)^2}{3\alpha_c c^2}, \quad \int_{\mathbb{R}} q_c'(x)^2 dx = \frac{8\alpha_c(c^2 - 1)^2}{15c^2}.$$

In view of (6.4) and (6.6), we have $\lambda_{1,0} := \sqrt{\frac{\langle \mathcal{L}_1(0)\zeta_1, \zeta_2^* \rangle}{-\langle \zeta_2, \zeta_1^* \rangle}} > 0$.

Since

$$\begin{aligned}
\partial_{\lambda_1} F_1(\lambda_{1,0}, \gamma_0, 0) &= 2\lambda_{1,0} \langle \zeta_2, \zeta_1^* \rangle, \quad \partial_{\gamma} F_1(\lambda_{1,0}, \gamma_0, 0) = \langle \zeta_1, \zeta_1^* \rangle \neq 0, \\
\partial_{\lambda_1} F_2(\lambda_{1,0}, \gamma_0, 0) &= 2\lambda_{1,0} \langle \zeta_2, \zeta_2^* \rangle \neq 0, \quad \partial_{\gamma} F_2(\lambda_{1,0}, \gamma_0, 0) = \langle \zeta_1, \zeta_2^* \rangle = 0,
\end{aligned}$$

it follows from the implicit function theorem that there exists an $\eta_0 > 0$, $\lambda_1(\eta)$, $\gamma(\eta) \in C^1([-\eta_0, \eta_0])$ such that $\lambda_1(0) = \lambda_{1,0}$, $\gamma(0) = \gamma_0$ and $F_k(\lambda_1(\eta), \gamma(\eta), \eta) = 0$ for $\eta \in [-\eta_0, \eta_0]$ and $k = 1, 2$. Moreover, we have

$$\lambda_1'(0) = -\frac{\partial_{\eta} F_2(\lambda_{1,0}, \gamma_0, 0)}{\partial_{\lambda_1} F_2(\lambda_{1,0}, \gamma_0, 0)} = \frac{i}{2} \left(\gamma_0 - \frac{\langle \mathcal{L}_1(0)\zeta_2, \zeta_2^* \rangle}{\langle \zeta_2, \zeta_2^* \rangle} \right) =: i\lambda_{2,0},$$

and $\lambda(\eta) = i\lambda_{1,0}\eta - \lambda_{2,0}\eta^2 + O(\eta^3)$. Thus we prove (2.4) and (2.5).

To obtain the asymptotic expansion of $\zeta^*(\eta)$, let $\tilde{v}_2(z, \eta) = \zeta(-z, -\eta) \cdot {}^t(1, 0)$, where \cdot denotes the inner product in \mathbb{C}^2 and

$$\zeta^*(\eta) = c \begin{pmatrix} (\lambda(-\eta) + c\partial_z)B(\eta)\tilde{v}_2(z, \eta) + v_{2,c}(\eta)^*\tilde{v}_2(z, \eta) \\ B(\eta)\tilde{v}_2(z, \eta) \end{pmatrix}.$$

As in the proof of Lemma 3.4, we have $\mathcal{L}(\eta)\zeta^*(\eta) = \lambda(-\eta)\zeta^*(\eta)$. Since

$$\lambda(-\eta) = -i\lambda_1\eta - \lambda_2\eta^2 + O(\eta^3),$$

$$\tilde{v}_2(\cdot, \eta) = q_c - i\lambda_1\eta \int_{-\infty}^z \partial_c q_c(z_1) dz_1 + O(\eta^2) \quad \text{in } H_{-\alpha}^k(\mathbb{R}) \text{ for any } k \geq 0,$$

we have (2.6). We can show (2.7) in the same way as the proof of Lemmas 3.3 and 3.4.

Finally, we will prove $\lambda_{2,0} > 0$. By (6.4) and the definition of $\lambda_{2,0}$,

$$(6.8) \quad \frac{d}{dc} E(q_c, r_c) \lambda_{2,0} = -\langle \mathcal{L}_1(0)\zeta_1, \zeta_1^* \rangle - \langle \mathcal{L}_1(0)\zeta_2, \zeta_2^* \rangle - \lambda_{1,0}^2 \langle \zeta_2, \zeta_1^* \rangle.$$

We have

$$\begin{aligned} \mathcal{L}_1(0)\zeta_2 = & B_0^{-1} \begin{pmatrix} 0 \\ A_0\partial_z^{-1}\partial_c q_c + (A_0 - B_0)B_0^{-1}\partial_z^{-1}\partial_c q_c \end{pmatrix} \\ & + bB_0^{-2} \begin{pmatrix} 0 \\ 3c\partial_z(q_c\partial_c q_c) + \frac{3}{2}\partial_z(q_c^2) \end{pmatrix} + B_0^{-1} \begin{pmatrix} 0 \\ cq_c\partial_z^{-1}\partial_c q_c \end{pmatrix}. \end{aligned}$$

Using the fact that q_c and $\partial_c q_c$ are even, q_c' is odd and $B_0^{-1}f$ retains the parity of f , we have

$$(6.9) \quad \langle \mathcal{L}_1(0)\zeta_2, \zeta_2^* \rangle = c\langle \partial_z^{-1}\partial_c q_c, q_c \rangle + c^2\langle \partial_z^{-1}\partial_c q_c, q_c^2 \rangle = \frac{c}{3}(2c^2 + 1)\langle \partial_z^{-1}\partial_c q_c, q_c \rangle.$$

In the last line, we use $(A_0 - c^2 B_0)q_c + \frac{3}{2}q_c^2 = 0$. Analogously, we have

$$(6.10) \quad \langle \mathcal{L}_1(0)\zeta_1, \zeta_1^* \rangle = c\langle q_c, (\partial_z^{-1})^* \partial_c q_c \rangle + c^2\langle q_c^2, (\partial_z^{-1})^* \partial_c q_c \rangle = \frac{c}{3}(2c^2 + 1)\langle \partial_z^{-1}\partial_c q_c, q_c \rangle,$$

where $(\partial_z^{-1})^* f = -\int_{-\infty}^z f(z_1) dz_1$. By integration by parts,

$$(6.11) \quad \langle \partial_z^{-1}\partial_c q_c, q_c \rangle = -\frac{1}{4}\frac{d}{dc} \left(\int_{\mathbb{R}} q_c dz \right)^2 = -4\frac{d}{dc} \frac{(c^2 - 1)(bc^2 - a)}{c^2}.$$

Combining (6.8)–(6.11) with (6.4)–(6.6), we have

$$\begin{aligned} (6.12) \quad \lambda_{2,0}\frac{d}{dc}E(q_c, r_c) = & -\lambda_{1,0}^2\langle \zeta_2, \zeta_1^* \rangle - \frac{16}{3}\frac{(1 + 2c^2)}{c^2}(bc^4 - a) \\ = & 32\frac{(b\rho^2 - a)}{3d(c)}n(c), \end{aligned}$$

where

$$(6.13) \quad d(c) = 6a^2 + (3a^2 - 9ab)c^2 + (6a^2 + 2b^2 - 2ab)c^4 + (b^2 - 19ab)c^6 + 12b^2c^8,$$

$$(6.14) \quad n(c) = 7a^2 - ba + (4a^2 - 10ba)c^2 + (3b^2 + 4a^2 - 7ab)c^4 + 6b(b - 2a)c^6 + 6b^2c^8.$$

To show that $n(c) > 0$ for all $c > 1$ and $b > a > 0$, we set $\rho = c^2$ and differentiate $n(\rho)$ twice to get

$$n'(\rho) = 4a^2 - 10ab + 2(3b^2 + 4a^2 - 7ab)\rho + 18(b^2 - 2ab)\rho^2 + 24b^2\rho^3,$$

and

$$n''(\rho) = 2(3b^2 + 4a^2 - 7ab) + 36b^2\rho + 72b(b\rho^2 - a\rho) > 0, \quad \forall \rho > 1.$$

Since $n'(1) = 12(b - a)^2 + 36(b - a)b > 0$, $n'(\rho) > 0$ for all $\rho > 1$. Since $n(1) = 15(b - a)^2 > 0$, thus, $n(\rho) > 0$ for all $\rho > 1$. In the same way, to show that $d(\rho) > 0$ for all $\rho > 1$ and $b > a > 0$, we set $\rho = c^2$ and differentiate $d(\rho)$ to obtain

$$d'(\rho) = (3a^2 - 9ab) + 2(6a^2 + 2b^2 - 2ab)\rho + 3(b^2 - 19ab)\rho^2 + 48b^2\rho^3,$$

and

$$d''(\rho) = 12a^2 + 4b(b - a) + 6b^2\rho + b\rho(-114a + 144b\rho) > 0, \quad \forall \rho > 1.$$

Since $d'(1) = 15(b - a)^2 + 40(b - a)b > 0$, $d'(\rho) > 0$ for all $\rho > 1$. Since $d(1) = 15(b - a)^2 > 0$, thus, $d(\rho) > 0$ for all $\rho > 1$.

Since $\frac{d}{dc}E(q_c, r_c) > 0$, $d(c) > 0$ and $n(c) > 0$ for $c > 1$, we conclude from (6.12) that $\lambda_{2,0} > 0$. This completes the proof of Theorem 2.1.

7. PROOF OF COROLLARY 2.2

The Gearhart-Prüss theorem [15, 37] tells us the semigroup estimate (2.8) follows from uniform boundedness of $(\lambda - \mathcal{L})^{-1}\mathcal{Q}(\eta_0)$ in a stable half plane. Let $\Omega = \{\lambda \mid \Re \lambda \geq -\beta'\}$. Applying [37, Corollary 4] to a Hilbert space $\mathcal{Q}(\eta_0)X$, we have (2.8) provided $(\lambda - \mathcal{L})^{-1}\mathcal{Q}(\eta_0)$ is uniformly bounded in Ω . Thus to prove Theorem 2.2, it suffices to show the following.

Lemma 7.1. *Let $c > 1$ and $\alpha \in (0, \alpha_c)$. Assume (H) for $\beta \in (0, \alpha(c-1)/2)$ and an $\eta_0 > 0$. Then for any $\beta' < \beta$,*

$$(7.1) \quad \sup_{\lambda \in \Omega} \|(\lambda - \mathcal{L})^{-1}\mathcal{Q}(\eta_0)\|_{B(X)} < \infty.$$

Proof. By (H), the restricted resolvent $(\lambda - \mathcal{L})^{-1}\mathcal{Q}(\eta_0)$ is uniformly bounded on any compact subset of Ω . Thus by Lemma 4.1 and (5.3), we have (7.1) provided

$$(7.2) \quad \sup_{\lambda \in \Omega, |\lambda| \geq K_1} \|(\lambda - \mathcal{L}_0)^{-1}V\|_{B(X)} \leq \frac{1}{2}$$

for sufficiently large K_1 . To prove (7.2), we apply the argument for the 1-dimensional Benney-Luke equation [30] for low frequencies in y and use the argument in §5.1 for high frequencies in y .

Let $K_2 > 0$, χ be the characteristic function of $[-K_2, K_2]$ and $\tilde{\chi}(\eta) = 1 - \chi(\eta)$ for $\eta \in \mathbb{R}$. First, we will show that

$$(7.3) \quad (\lambda - \mathcal{L}_0)^{-1}V\chi(D_y) = \begin{pmatrix} r_{11}(\lambda) & r_{12}(\lambda) \\ r_{21}(\lambda) & r_{22}(\lambda) \end{pmatrix} \chi(D_y) \rightarrow 0 \quad \text{uniformly as } \lambda \rightarrow \infty \text{ with } \lambda \in \Omega.$$

By (5.6) and (5.8),

$$\begin{aligned} r_{11}(\lambda)\chi(D_y) &= \frac{ic}{2}S(D)^{-1}\{(\lambda - \lambda_+(D))^{-1} - (\lambda - \lambda_-(D))^{-1}\}\mu(D)B^{-1}q_c\chi(D_y) \\ &\quad - c(\lambda - \lambda_+(D))^{-1}(\lambda - \lambda_-(D))^{-1}B^{-1}q_c''\chi(D_y). \end{aligned}$$

By the Plancherel theorem,

$$\|(\lambda - \lambda_{\pm}(D))^{-1}\chi(D_y)f\|_{L_{\alpha}^2(\mathbb{R}^2)} = \left\| \frac{\hat{f}(\xi + i\alpha, \eta)}{\lambda - \lambda_{\pm}(\xi + i\alpha, \eta)} \right\|_{L^2(\mathbb{R}_{\xi} \times [-K_2, K_2]}.$$

In view of (4.9) and (4.11), we have $\lim_{\lambda \in \Omega, \lambda \rightarrow \infty} \|(\lambda - \lambda_{\pm}(D))^{-1}f\|_{L_{\alpha}^2(\mathbb{R}^2)} = 0$ for any $f \in L_{\alpha}^2$ thanks to the dominated convergence theorem. Thus we prove $(\lambda - \lambda_{\pm}(D))^{-1} \rightarrow 0$ strongly as $\lambda \rightarrow \infty$ with $\lambda \in \Omega$. Since $\mu(D)B^{-1}q_c\chi(D_y)$, $B^{-1}q_c''\chi(D_y) : H_{\alpha}^1 \rightarrow H_{\alpha}^1$ are compact, we see that $\lim_{\lambda \in \Omega, \lambda \rightarrow \infty} \|r_{11}(\lambda)\|_{B(H_{\alpha}^1)} = 0$ as in [30, p.265]. We can prove

$$\lim_{\lambda \in \Omega, \lambda \rightarrow \infty} (\|r_{12}(\lambda)\chi(D_y)\|_{B(L_{\alpha}^2, H_{\alpha}^1)} + \|r_{21}(\lambda)\chi(D_y)\|_{B(H_{\alpha}^1, L_{\alpha}^2)} + \|r_{22}(\lambda)\chi(D_y)\|_{B(L_{\alpha}^2)}) = 0$$

in exactly the same way.

By Lemma 4.6 and the definition of $r_{ij}(\lambda)$,

$$(7.4) \quad \begin{aligned} &\|r_{ij}(\lambda)\tilde{\chi}(D_y)\|_{B(H_{\alpha}^{2-j}, H_{\alpha}^{2-i})} \\ &\leq \|\tilde{\chi}(D_y)\mu(D)B^{-1}\|_{B(L_{\alpha}^2)} + \|\tilde{\chi}(D_y)B^{-1}\|_{B(L_{\alpha}^2)} \rightarrow 0 \quad \text{as } K_2 \rightarrow \infty. \end{aligned}$$

Combining (7.3) and (7.4), we have (7.2). Thus we complete the proof. \square

8. PROOF OF THEOREM 2.3

Let

$$g(z, \eta) = \left(1 + i \frac{\Re \langle \zeta(\cdot, \eta), \zeta^*(\cdot, \eta) \rangle}{\Im \langle \zeta(\cdot, \eta), \zeta^*(\cdot, \eta) \rangle}\right) \zeta(x, \eta), \quad g^*(x, \eta) = \zeta^*(x, \eta),$$

and define $g_k(x, \eta)$ and $g_k^*(x, \eta)$ ($k = 1, 2$) by (3.42) and (3.43) as in Section 3. By (2.7), we have for $\eta \in [-\eta_0, \eta_0]$, $z \in \mathbb{R}$ and $k = 1, 2$,

$$\overline{g(z, \eta)} = g(z, -\eta), \quad \overline{g^*(x, \eta)} = g^*(x, -\eta),$$

$$\kappa(\eta) := \frac{1}{2} \Im \langle g(\cdot, \eta), g^*(\cdot, \eta) \rangle \quad \text{is odd},$$

and $g_k(z, \eta)$ and $g_k^*(z, \eta)$ are real valued and even in η . Moreover,

$$\langle g_j(\cdot, \eta), g_k^*(\cdot, \eta) \rangle = \delta_{jk} \quad \text{for } j, k = 1, 2.$$

By Theorem 2.1 and (6.4),

$$\begin{aligned} \langle \zeta(\cdot, \eta), \zeta^*(\cdot, \eta) \rangle &= \langle \zeta_1, \zeta_2^* \rangle + i\lambda_1 \eta \{ \langle \zeta_2, \zeta_2^* \rangle + \langle \zeta_1, \zeta_1^* \rangle \} + O(\eta^2) \\ &= 2i\kappa_1 \eta + O(\eta^2), \end{aligned}$$

and

$$\begin{aligned} (8.1) \quad \kappa(\eta) &= \frac{1}{2} \Im \langle \zeta(\cdot, \eta), \zeta^*(\cdot, \eta) \rangle \left\{ 1 + \left(\frac{\Re \langle \zeta(\cdot, \eta), \zeta^*(\cdot, \eta) \rangle}{\Im \langle \zeta(\cdot, \eta), \zeta^*(\cdot, \eta) \rangle} \right)^2 \right\} \\ &= \kappa_1 \eta + O(\eta^3). \end{aligned}$$

Let $\vec{\Phi}(t) = (\Phi(t), \Psi(t))$ be a solution of (2.2) with $\vec{\Phi}(0) = (\Phi_0, \Psi_0)$ and

$$c_k(t, \eta) = \left\langle \mathcal{F}_y \vec{\Phi}(t, \cdot, \eta), g_k^*(\cdot, \eta) \right\rangle \quad \text{for } \eta \in [-\eta_0, \eta_0] \text{ and } k = 1, 2.$$

Then

$$\vec{\Phi}(t) = \frac{1}{\sqrt{2\pi}} \sum_{k=1,2} \int_{-\eta_0}^{\eta_0} c_k(t, \eta) g_k(z, y) e^{iy\eta} d\eta.$$

By Remark 3.1,

$$\partial_t \begin{pmatrix} c_1(t, \eta) \\ c_2(t, \eta) \end{pmatrix} = \begin{pmatrix} \langle \mathcal{L}(\eta) \mathcal{F}_y \vec{\Phi}(t, \cdot, \eta), g_1^*(\cdot, \eta) \rangle \\ \langle \mathcal{L}(\eta) \mathcal{F}_y \vec{\Phi}(t, \cdot, \eta), g_2^*(\cdot, \eta) \rangle \end{pmatrix} = \mathcal{A}(\eta) \begin{pmatrix} c_1(t, \eta) \\ c_2(t, \eta) \end{pmatrix},$$

where

$$\mathcal{A}(\eta) = \begin{pmatrix} \Re \lambda(\eta) & \frac{\Im \lambda(\eta)}{\kappa(\eta)} \\ -\kappa(\eta) \Im \lambda(\eta) & \Re \lambda(\eta) \end{pmatrix}.$$

Let $e(t, \eta) = |\kappa(\eta) c_1(t, \eta)|^2 + |c_2(t, \eta)|^2$. Then $e(t, \eta) = e^{2t \Re \lambda(\eta)} e(0, \eta)$ and

$$\begin{aligned} (8.2) \quad & \|\eta^{k+1} c_1(t, \eta)\|_{L^2(-\eta_0, \eta_0)}^2 + \|\eta^k c_2(t, \eta)\|_{L^2(-\eta_0, \eta_0)}^2 \\ & \lesssim \int_{-\eta_0}^{\eta_0} \eta^{2k} e(t, \eta) d\eta \\ & \lesssim (1+t)^{-k} \{ \|\eta^{k+1} c_1(0, \eta)\|_{L^2(-\eta_0, \eta_0)}^2 + \|\eta^k c_2(0, \eta)\|_{L^2(-\eta_0, \eta_0)}^2 \} \\ & \lesssim (1+t)^{-k} (\|\Phi_0\|_{L_\alpha^2(\mathbb{R}^2)} + \|\Psi_0\|_{L_\alpha^2(\mathbb{R}^2)}). \end{aligned}$$

because $\kappa(\eta) = \kappa_1\eta + O(\eta^3)$ with $\kappa_1 \neq 0$ and $\Re\lambda(\eta) = -\lambda_2\eta^2 + O(\eta^4)$ with $\lambda_2 > 0$. Since $\kappa(\eta)$ and $\Im\lambda(\eta)$ are odd and $\Re\lambda(\eta)$ is even, it follows from Theorem 2.1 and (8.1) that

$$\mathcal{A}(\eta) = \mathcal{A}_0(\eta) + \begin{pmatrix} O(\eta^4) & O(\eta^2) \\ O(\eta^4) & O(\eta^4) \end{pmatrix}, \quad \mathcal{A}_0(\eta) = \begin{pmatrix} -\lambda_2\eta^2 & \frac{\lambda_1}{\kappa_1} \\ -\lambda_1\kappa_1\eta^2 & -\lambda_2\eta^2 \end{pmatrix}.$$

By the variation of the constants formula,

$$\begin{pmatrix} c_1(t, \eta) \\ c_2(t, \eta) \end{pmatrix} = e^{t\mathcal{A}_0(\eta)} \begin{pmatrix} c_1(0, \eta) \\ c_2(0, \eta) \end{pmatrix} - \int_0^t e^{(t-s)\mathcal{A}_0(\eta)} (\mathcal{A}(\eta) - \mathcal{A}_0(\eta)) \begin{pmatrix} c_1(s, \eta) \\ c_2(s, \eta) \end{pmatrix} ds,$$

where $e^{t\mathcal{A}_0(\eta)} = e^{-t\lambda_2\eta^2} \begin{pmatrix} \cos t\lambda_1\eta & \frac{\sin t\lambda_1\eta}{\kappa_1\eta} \\ -\kappa_1\eta \sin t\lambda_1\eta & \cos t\lambda_1\eta \end{pmatrix}$. Using (8.2), we have for $k = 0$ and 1,

$$\begin{aligned} & \left\| \eta^k \left\{ c_1(t, \eta) - e^{-t\lambda_2\eta^2} \frac{\sin t\lambda_1\eta}{\kappa_1\eta} c_2(0, \eta) \right\} \right\|_{L^2(-\eta_0, \eta_0)} \\ & \lesssim \| \eta e^{-t\lambda_2\eta^2} c_1(0, \eta) \|_{L^2(-\eta_0, \eta_0)} + \sum_{j=1,2} \int_0^t \| \eta^{4+k-j} e^{-(t-s)\lambda_2\eta^2} c_j(s, \eta) \|_{L^2(-\eta_0, \eta_0)} ds \\ (8.3) \quad & \lesssim (1+t)^{-k/2} \| c_1(0, \eta) \|_{L^2(-\eta_0, \eta_0)} \\ & \quad + \int_0^t (1+t-s)^{-3/4} (1+s)^{-(2k+1)/4} ds \| \eta^{\frac{5}{2}+k-j} e(0, \eta) \|_{L^2(-\eta_0, \eta_0)} \\ & \lesssim (1+t)^{-k/2} (\| \Phi_0 \|_{L_\alpha^2(\mathbb{R}^2)} + \| \Psi_0 \|_{L_\alpha^2(\mathbb{R}^2)}). \end{aligned}$$

Since $f(y) = \langle \vec{\Phi}(0, \cdot, y), \zeta_2^* \rangle$ and $\|g_2^*(\cdot, \eta) - \zeta_2^*\|_{L_{-\alpha}^2(\mathbb{R})} = O(\eta^2)$, we have

$$\begin{aligned} |c_2(0, \eta) - \hat{f}(\eta)| & \leq \| \mathcal{F}_y \vec{\Phi}(0, \cdot, \eta) \|_{L_\alpha^2(\mathbb{R})} \| g_2^*(\cdot, \eta) - \zeta_2^* \|_{L_{-\alpha}^2(\mathbb{R})} \\ & \lesssim \eta^2 (\| \mathcal{F}_y \Phi_0(\cdot, \eta) \|_{L_\alpha^2(\mathbb{R})} + \| \mathcal{F}_y \Psi_0(\cdot, \eta) \|_{L_\alpha^2(\mathbb{R})}), \end{aligned}$$

and

$$(8.4) \quad \left\| e^{-t\lambda_2\eta^2} \frac{\sin t\lambda_1\eta}{\kappa_1\eta} \{ c_2(0, \eta) - \hat{f}(\eta) \} \right\|_{L^2(-\eta_0, \eta_0)} \lesssim (1+t)^{-1/2} (\| \Phi_0 \|_{L_\alpha^2(\mathbb{R}^2)} + \| \Psi_0 \|_{L_\alpha^2(\mathbb{R}^2)}).$$

Combining (8.3) and (8.4) with $\|g_1(\cdot, \eta) - \zeta_1\|_{L_\alpha^2(\mathbb{R})} = O(\eta^2)$, we have for $k = 0$ and 1,

$$\begin{aligned} & \left\| \eta^k \left\{ c_1(t, \eta) g_1(\cdot, \eta) - e^{-t\lambda_2\eta^2} \frac{\sin t\lambda_1\eta}{\kappa_1\eta} \hat{f}(\eta) \zeta_1 \right\} \right\|_{L^2([-\eta_0, \eta_0]; L_\alpha^2(\mathbb{R}_z))} \\ & \lesssim \| \eta^{k+2} c_1(t, \eta) \|_{L^2(-\eta_0, \eta_0)} \sup_{0 < |\eta| \leq \eta_0} \eta^{-2} \| g_1(\cdot, \eta) - \zeta_1 \|_{L_\alpha^2(\mathbb{R})} \\ (8.5) \quad & \quad + \left\| \eta^k \left\{ c_1(t, \eta) - e^{-t\lambda_2\eta^2} \frac{\sin t\lambda_1\eta}{\kappa_1\eta} \hat{f}(\eta) \right\} \right\|_{L^2(-\eta_0, \eta_0)} \| \zeta_1 \|_{L_\alpha^2(\mathbb{R})} \\ & \lesssim (1+t)^{-k/2} (\| \Phi_0 \|_{L_\alpha^2(\mathbb{R}^2)} + \| \Psi_0 \|_{L_\alpha^2(\mathbb{R}^2)}). \end{aligned}$$

Since $\| \hat{f} \|_{L^2} = \| f \|_{L^2} \lesssim \| \Phi_0 \|_{L_\alpha^2(\mathbb{R}^2)} + \| \Psi_0 \|_{L_\alpha^2(\mathbb{R}^2)}$,

$$(8.6) \quad \left\| e^{-t\lambda_2\eta^2} \frac{\sin t\lambda_1\eta}{\kappa_1\eta} \hat{f}(\eta) \zeta_1(z) \right\|_{L^2(|\eta| \geq \eta_0; L_\alpha^2(\mathbb{R}_z))} \lesssim e^{-t\lambda_2\eta_0^2} (\| \Phi_0 \|_{L_\alpha^2(\mathbb{R}^2)} + \| \Psi_0 \|_{L_\alpha^2(\mathbb{R}^2)}).$$

Using the Plancherel theorem, (8.2), (8.5) and (8.6), we have

$$\begin{aligned}
& \|\partial_y^j \{\mathcal{P}(\eta_0) \vec{\Phi}(t) - (H_t * W_t * f)(y) \zeta_1(z)\}\|_X \\
& \lesssim \left\| \eta^j \{c_1(t, \eta) g_1(z, \eta) - e^{-t\lambda_2 \eta^2} \frac{\sin t \lambda_1 \eta}{\kappa_1 \eta} \hat{f}(\eta) \zeta_1(z)\} \right\|_{L^2([- \eta_0, \eta_0]; H_\alpha^1(\mathbb{R}_z) \times L_\alpha^2(\mathbb{R}_z))} \\
& \quad + \|\eta^j c_2(t, \eta)\|_{L^2(-\eta_0, \eta_0)} + \|\eta^j e^{-t\lambda_2 \eta^2} \frac{\sin t \lambda_1 \eta}{\kappa_1 \eta} \hat{f}(\eta)\|_{L_\alpha^2(|\eta| \geq \eta_0)} \\
& \lesssim (1+t)^{-j/2} (\|\Phi_0\|_{L_\alpha^2(\mathbb{R}^2)} + \|\Psi_0\|_{L_\alpha^2(\mathbb{R}^2)}).
\end{aligned}$$

By Theorem 2.2,

$$\|\mathcal{Q}(\eta_0) \vec{\Phi}(t)\|_{H_\alpha^2(\mathbb{R}^2) \times H_\alpha^1(\mathbb{R}^2)} \lesssim e^{-\beta' t} (\|\Phi_0\|_{H_\alpha^2(\mathbb{R}^2)} + \|\Psi_0\|_{H_\alpha^1(\mathbb{R}^2)}).$$

Combining the above, we obtain for $j = 0, 1$,

$$\begin{aligned}
& \left\| \text{diag}(\partial_z^j, 1) \{\vec{\Phi}(t, z, y) - (H_t * W_t * f)(y) \zeta_1(z)\} \right\|_{L_\alpha^2(\mathbb{R}_z) L^\infty(\mathbb{R}_y)} \\
& \lesssim (1+t)^{-1/4} (\|\Psi_0\|_{H_\alpha^2(\mathbb{R}^2)} + \|\Psi_0\|_{H_\alpha^1(\mathbb{R}^2)}).
\end{aligned}$$

This completes the proof of Theorem 2.3.

APPENDIX A. MISCELLANEOUS ESTIMATES OF OPERATOR NORMS

In this section, we collect estimates of the norm of operators.

A solitary wave profile $q_c(x)$ is similar to KdV 1-solitons provided c is close to 1. In view of (2.10), we have the following estimates on derivatives of q_c .

Claim A.1. *Let $c = \sqrt{1 + \epsilon^2}$, $\alpha = \hat{\alpha}\epsilon$ and $\hat{\alpha} \in (0, \hat{\alpha}_0/2)$. There exists positive constants ϵ_0 and C such that*

$$\|\partial_z^i \partial_c^j q_c\|_{B(L_\alpha^2(\mathbb{R}))} \leq C \epsilon^{2+i-2j}. \quad \text{for } \epsilon \in (0, \epsilon_0) \text{ and } i, j \in \mathbb{Z}_{\geq 0}.$$

Next, we collect estimates of ∂_z , $\mu(D)$, $S(D)$ and B^{-1} .

Claim A.2. *Let $\hat{\alpha} > 0$ and $\alpha = \hat{\alpha}\epsilon$. There exists a positive constants ϵ_0 such that if $\epsilon \in (0, \epsilon_0)$,*

$$(A.1) \quad \|\partial_z^{-1}\|_{B(L_\alpha^2)} \leq \alpha^{-1},$$

$$(A.2) \quad \|\mu(D)^{-1}\|_{B(Y)} \leq \sqrt{2} \alpha^{-1}, \quad \|\partial_z \mu(D)^{-1}\|_{B(Y)} \leq \sqrt{2},$$

$$(A.3) \quad \|\partial_z\|_{B(Y_{low})} \leq (K + \hat{\alpha})\epsilon, \quad \|\mu(D)^j\|_{B(Y_{low})} \leq \{2(K + \hat{\alpha})\epsilon\}^j \text{ for } j \in \mathbb{N},$$

$$(A.4) \quad \|i\partial_z \mu(D)^{-1} + I\|_{B(Y_{low})} = O(K^4 \epsilon^2).$$

Proof. By (3.5),

$$\|\partial_z^{-1}\|_{B(L_\alpha^2)} = \sup_{\xi \in \mathbb{R}} \left| \frac{1}{\xi + i\alpha} \right| \leq \alpha^{-1},$$

$$\|\partial_z^j \mu(D)^{-1}\|_{B(Y)} = \sup_{(\xi, \eta) \in \mathbb{R} \times [-K(K+\hat{\alpha})\epsilon^2, K(K+\hat{\alpha})\epsilon^2]} \frac{|\xi + i\alpha|^j}{|\mu(\xi + i\alpha, \eta)|}.$$

If $\eta \in [-K^2\epsilon^2, K^2\epsilon^2]$ and ϵ is sufficiently small, then $\eta^2 \leq \alpha^2/2$ and

$$|\mu(\xi + i\alpha, \eta)|^4 = (\xi^2 + \alpha^2 - \eta^2)^2 + 4\xi^2\eta^2 \geq \frac{1}{4}(\xi^2 + \alpha^2)^2.$$

Combining the above, we have (A.2).

Since $\text{supp } \hat{f}(\xi + i\alpha, \eta) \subset \tilde{A}_{low}$ for $f \in Y_{low}$, we have (A.3) and

$$\|i\partial_z \mu(D)^{-1} + I\|_{B(Y_{low})} = \sup_{(\xi, \eta) \in \tilde{A}_{low}} \left| \left\{ 1 + \frac{\eta^2}{(\xi + i\alpha)^2} \right\}^{-1/2} - 1 \right| = O(K^4\epsilon^2).$$

Thus we complete the proof. \square

Claim A.3. *Let $\hat{\alpha} > 0$ and $\alpha = \hat{\alpha}\epsilon$. There exists positive constants C and ϵ_0 such that for any $\epsilon \in (0, \epsilon_0)$,*

$$(A.5) \quad \|S(D)\|_{B(L_\alpha^2)} + \|S(D)^{-1}\|_{B(L_\alpha^2)} \leq C,$$

$$(A.6) \quad \|\partial_z^j B^{-1}\|_{B(L_\alpha^2, H_\alpha^{2-j})} + \|\mu_j(D)^j B^{-1}\|_{B(L_\alpha^2, H_\alpha^{2-j})} \leq C \quad \text{for } j = 0, 1, 2,$$

$$(A.7) \quad \|[B, \partial_z^j q_c]\|_{B(H_\alpha^1, L_\alpha^2)} \leq C\epsilon^{j+3},$$

$$(A.8) \quad \|B^{-1} - I\|_{B(Y_{low})} + \|S(D) - I\|_{B(Y_{low})} + \|S^{-1}(D) - I\|_{B(Y_{low})} \leq CK^2\epsilon^2.$$

Proof. We can prove (A.5)–(A.7) in the same way as Lemmas 7.2 and 7.4 in [30]. Since

$$B(\xi + i\alpha, \eta) = 1 + b\{(\xi + i\alpha)^2 + \eta^2\} = 1 + O(K^2\epsilon^2) \quad \text{for } (\xi, \eta) \in \tilde{A}_{low},$$

we have

$$\|B^{-1} - I\|_{Y_{low}} = \sup_{(\xi, \eta) \in \tilde{A}_{low}} |B^{-1}(\xi + i\alpha, \eta) - 1| = O(K^2\epsilon^2).$$

Similarly, we have $\|S(D) - I\|_{B(Y_{low})} + \|S^{-1}(D) - I\|_{B(Y_{low})} = O(K^2\epsilon^2)$ from (4.31). \square

Next, we will estimates the operator norms of a_1 and a_2 .

Claim A.4. *Let $\hat{\alpha} \in (0, \hat{\alpha}_0/2)$ and $c = \sqrt{1 + \epsilon^2}$. There exists an $\epsilon_0 > 0$ such that if $\epsilon \in (0, \epsilon_0)$ and $\alpha = \epsilon\hat{\alpha}$, then*

$$(A.9) \quad \|a_i\|_{B(Y)} = O(\epsilon^2) \quad \text{for } i = 1, 2,$$

$$(A.10) \quad \|a_i \rho_z(D_z)\|_{B(Y)} + \|\rho_z(D_z) a_i\|_{B(Y)} = O(K\epsilon^3) \quad \text{for } i = 1, 2,$$

$$(A.11) \quad \left\| \rho_{KP}(D) \left\{ a_{2,\epsilon} + \frac{3}{2} \partial_z(\theta_0 \cdot) \right\} \rho_{KP}(D) \right\|_{B(L_\alpha^2(\mathbb{R}^2))} = O(K^5\epsilon^2).$$

Proof. By Claims A.1–A.3, (5.9) and (5.23), we have

$$\begin{aligned} & \|B^{-1} v_{1,c} \mu(D)^{-1}\|_{B(L_\alpha^2)} + \|B^{-1} v_{2,c}\|_{B(L_\alpha^2)} = O(\epsilon^2), \\ & \|B^{-1} v_{1,c} \mu(D)^{-1} \rho_z(D_z)\|_{B(Y)} + \|B^{-1} v_{2,c} \rho_z(D_z)\|_{B(L_\alpha^2)} = O(K\epsilon^3), \\ & \|\rho_z(D_z) B^{-1} v_{1,c} \mu(D)^{-1}\|_{B(Y)} + \|\rho_z(D_z) B^{-1} v_{2,c}\|_{B(L_\alpha^2)} = O(K\epsilon^3). \end{aligned}$$

Combining the above with (A.5), we have (A.9) and (A.10).

Finally, we will prove (A.11). By (A.8),

$$\begin{aligned} & \|\rho_z(D_z)\{2a_2 + 3c(q\partial_z + q'_c)\}\rho_z(D_z)\|_{B(Y)} \\ & \leq \|\{iv_{1,c}\mu(D)^{-1} - c(q_c\partial_z + 2q'_c)\}\rho_z(D_z)\|_{B(Y)} \\ & \quad + O(K^2\epsilon^2(\|\rho_z(D_z)v_{1,c}\mu(D)^{-1}\rho_z(D_z)\|_{B(Y)} + \|\rho_z(D_z)v_{2,c}\rho_z(D_z)\|_{B(Y)})) . \end{aligned}$$

Claims A.1 and A.2 imply

$$\|\rho_z(D_z)v_{1,c}\mu(D)^{-1}\rho_z(D_z)\|_{B(Y)} + \|\rho_z(D_z)v_{2,c}\rho_z(D_z)\|_{B(Y)} = O(K\epsilon^3),$$

and

$$\begin{aligned} & \|\{iv_{1,c}\mu(D)^{-1} - c(q_c\partial_z + 2q'_c)\}\rho_z(D_z)\|_{B(Y)} \\ & \lesssim \|q_c\|_{L^\infty}\|(i\mu(D) - \partial_z)\rho_z(D_z)\|_{B(Y)} + \|q'_c\|_{L^\infty}\|(i\partial_z\mu(D)^{-1} + I)\rho_z(D_z)\|_{B(Y)} \\ & \quad + (c-1)\|(q_c\partial_z + 2q'_c)\rho_z(D_z)\|_{B(Y)} \\ & \lesssim K^5\epsilon^5 . \end{aligned}$$

In the last inequality, we use the fact that $c = 1 + O(\epsilon^2)$. Combining the above with the fact that $\|\epsilon^{-2}q_c(\cdot/\epsilon) - \theta_0\|_{C^1} = O(\epsilon^2)$, we have (A.11). Thus we complete the proof. \square

Claim A.5.

$$(A.12) \quad \|\bar{r}_{ij}\|_{B(Y)} \lesssim K\epsilon^3 \quad \text{for } i, j = 1, 2,$$

$$(A.13) \quad \|\bar{r}_{22}\|_{B(Y)} \lesssim K^5\epsilon^5 .$$

Proof. By Lemma 5.2,

$$\Pi^{-1} = \begin{pmatrix} I & O \\ \epsilon_{21} & I + \epsilon_{22} \end{pmatrix}$$

with $\|\epsilon_{2j}\|_{B(L_\alpha^2(\mathbb{R}^2))} = O(K^{-1})$ and for ${}^t(\tilde{u}_1, \tilde{u}_2) \in \tilde{Z}$ and ${}^t(\bar{u}_1, \bar{u}_2) = \Pi({}^t(\tilde{u}_1, \tilde{u}_2))$,

$$\begin{aligned} & \begin{pmatrix} \bar{r}_{11} & \bar{r}_{12} \\ \bar{r}_{21} & \bar{r}_{22} \end{pmatrix} \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} = \left[\Pi, \begin{pmatrix} \lambda_+(D) + a_1 & a_2 \\ a_1 & \lambda_-(D) + a_2 \end{pmatrix} \right] \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} \\ & = \begin{pmatrix} -a_2\rho_z(D_z)E_\epsilon^{-1}\mathcal{P}_{KP}(\eta_0)E_\epsilon\rho_z(D_z)\tilde{u}_2 \\ \rho_z(D_z)E_\epsilon^{-1}\mathcal{P}_{KP}(\eta_0)E_\epsilon\rho_z(D_z)a_1\tilde{u}_1 - [\lambda_-(D) + a_2, \rho_z(D_z)E_\epsilon^{-1}\mathcal{P}_{KP}(\eta_0)E_\epsilon\rho_z(D_z)]\tilde{u}_2 \end{pmatrix} . \end{aligned}$$

Combining the above with Claim A.4, we have (A.12).

Next, we will prove (A.13) by using the KP-II approximation of $\lambda_{-, \epsilon}(D) + a_{2, \epsilon}$ in the low frequency regime. Since

$$\begin{aligned} \bar{r}_{22}\bar{u}_2 & = -[\lambda_-(D) + a_2, \rho_z(D_z)E_\epsilon^{-1}\mathcal{P}_{KP}(\eta_0)E_\epsilon\rho_z(D_z)]\tilde{u}_2 \\ & = -\epsilon^3 E_\epsilon^{-1}[\lambda_{-, \epsilon}(D) + a_{2, \epsilon}, \rho_z(\epsilon D_z)\mathcal{P}_{KP}(\eta_0)\rho_z(\epsilon D_z)]E_\epsilon\tilde{u}_2 , \end{aligned}$$

it follows from (4.25), (5.18) and (A.11),

$$\begin{aligned} & \|(\lambda_{-, \epsilon}(D) + a_{2, \epsilon})\rho_z(\epsilon D_z)g_{0, k}(\cdot, \eta) - \mathcal{L}_{KP}(\eta)g_{0, k}(\cdot, \eta)\|_{L_\alpha^2} \\ (A.14) \quad & + \|(\lambda_{-, \epsilon}(D) + a_{2, \epsilon})^*\rho_z(\epsilon D_z)g_{0, k}^*(\cdot, \eta) - \mathcal{L}_{KP}(\eta)^*g_{0, k}^*(\cdot, \eta)\|_{L_{-\alpha}^2} \\ & = O(K^8\epsilon^2) . \end{aligned}$$

Since $\mathcal{L}_{KP}\mathcal{P}_{KP}(\eta_0) = \mathcal{P}_{KP}(\eta_0)\mathcal{L}_{KP}$, we have (A.13) from (A.14). \square

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